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On Transmuted Flexible Weibull Extension Distribution with Applications to Different Lifetime Data Sets

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ABSTRACT

In this article, a three parameters transmuted version of the flexible Weibull extension distribution called transmuted flexible Weibull extension distribution is studied. The proposed model is very flexible and is able to model real phenomena with increasing, unimodal or modified unimodal failure rates. Explicit expressions for mathematical properties are derived. Maximum likelihood estimates and asymptotic confidence bounds for the unknown parameters of the model are also obtained. Four real data sets are analyzed in order to illustrate the flexibility of the proposed distribution.

Keywords: Flexible Weibull extension distribution, Modified unimodal failure rate, Order statistics, Moment generating function, Maximum likelihood estimation

1.   INTRODUCTION

In the family of aging distributions such as Exponential, Rayleigh, Pareto, Weibull, Lognormal and Gamma distributions etc. the Weibull distribution holds a key position and offers a wide range of applications in reliability and biological disciplines. The Weibull model was first applied by Waloddi Weibull for analyzing the breaking strength of materials, and so far it has been frequently used in various applied fields for modeling real phenomena. Some practical utilities of the Weibull model have been studied by Attardi et al. [2], Durham and Padgett [6] and Nadarajah and Kotz [13]. Due to its shape parameter, the Weibull model is a very flexible model and provides the characteristics of other ageing distributions such as Rayleigh and Exponential distributions. The Weibull model has monotonic hazard function and has been found very handy for modeling lifetime data with monotonic failure rates. But, it is inappropriate to use for modeling data with non-monotonic failure rates such as bathtub, upside down bathtub (unimodal) or modified upside down bathtub (modified unimodal) shaped failure rates. The characteristics of non-monotonic failure are quite common in biological studies and reliability engineering. For example, the behaviour of human mortality follows bathtub failure rate where initially the failure rate is very high, then gradually decrease followed by approximately a constant period and then gradually increases. The unimodal failure rate can be observed in biological studies where the failure rate reaches its maximum (peak point) after some finite period of time and then gradually decreases. The unimodal failure rate is very convenient for discovering the time period having maximum risk. The death rate of cancer patients is observed to follow modified unimodal failure rate. The unimodal failure rate has three phases, initially increases, then decreases, and again increases, for further about unimodal failure rate we call cf. Zajicek [21]. To obtain non-monotonic failure rate, the Weibull model has been modified and generalized by so many researchers, for example, additive Weibull (AW) distribution due to Xie and Lai [20], generalized flexible Weibull extension (GFWEx) distribution of Ahmad and Iqbal [1], transmuted Weibull (TW) distribution by Aryal and Tsokos [3], transmuted modified Weibull (TMW) distribution due to Khan and King [10], Kumaraswamy transmuted exponentiated additive Weibull (KTEAW) distribution of Nofal et al. [15] and transmuted additive Weibull (TAW) distribution studied by Elbatal and Aryal [7]. Bebbington et al. [5] proposed a very interesting modified form of Weibull model called flexible Weibull extension (FWEx) distribution. The FWEx distribution has two parameters and its cumulative distribution function (CDF) is given by

$$F(z;\theta,\lambda) = 1 - e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}, \quad z > 0, \quad (1)$$

where $\theta, \lambda > 0,$ are the model parameters. The probability density function (PDF) corresponding to (1) is

$$f(z;\theta,\lambda) = \left(\lambda + \frac{\theta}{z^2}\right) e^{\left(\lambda z - \frac{\theta}{z}\right)} e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}. \quad (2)$$

The Survival function (SF) of TFWEx random variable is

$$S(z;\theta,\lambda) = e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}},$$

with hazard function (HF),

$$h(z;\theta,\lambda) = \left(\lambda + \frac{\theta}{z^2}\right) e^{\left(\lambda z - \frac{\theta}{z}\right)}.$$

The FWEx distribution has non-monotonic HF. As we discussed earlier that lots of generalized forms of Weibull distribution have been proposed that has non-monotonic failure rate. But, many of these generalized forms of Weibull model does not have closed form of its CDF, SF and HF for example, Gamma Weibull (GW) distribution proposed by Stacy [16], beta inverse Weibull (BIW) distribution proposed by Hanook et al. [9] and beta modified Weibull (BMW) distribution due to Silva et al. [18]. Due to incomplete form of CDF, the estimation difficulties have increased. To address some of the problems that have been occurred with some modified forms of Weibull model; we propose a new model by generalizing the FWEx distribution using quadratic rank transmutation map (QRTM). The new model may be named as transmuted flexible Weibull extension (TFWEx) distribution, and possess a closed form of CDF allowing a very simple expression for SF and HF. The proposed model provides greater flexibility and modeling real phenomena with increasing and modified unimodal shaped failure rate.

A transmuted random variable has the CDF given by

$$G(z) = (1+\beta)F(z) - \beta(F(z))^2, \quad |\beta| \le 1, \quad (3)$$

here β represents the transmuted parameter, by putting $\beta = 0$, in (3) we get the CDF of the parent random variable, the transmuted density associating to (3) is given by

$$g(z) = f(z)(1+\beta - 2\beta F(z)). \quad (4)$$

Transmutation is a very convenient approach to extend any parent distribution. The method of QRTM has been applied by many different authors to extend the parent distribution for example, Aryal and Tsokos [3] studied transmuted Weibull (TW) distribution has the CDF given by

$$G_{TW}(z) = \left(1 - e^{-\gamma x^\alpha}\right)\left(1 + \beta e^{-\gamma x^\alpha}\right), \quad z, \alpha, \gamma > 0, \quad |\beta| \le 1.$$

Ashour and Eltehiwy [4] introduced transmuted exponentiated modified Weibull (TEMW) distribution defined by CDF

$$G_{TEMW}(z) = \left(1 - e^{-(\eta z + \gamma x^\alpha)}\right)^\theta \left(1 + \beta - \beta\left(1 - e^{-(\eta z + \gamma x^\alpha)}\right)^\theta\right),$$

where, $z, \alpha, \eta, \gamma, \theta > 0, \quad |\beta| \le 1$.

Elbatal and Aryal [7] proposed transmuted additive Weibull (TAW) distribution has the CDF given below

$$G_{TAW}(z) = \left(1 - e^{-\eta z^\theta - \gamma x^\alpha}\right)\left(1 + \beta e^{-\eta z^\theta - \gamma x^\alpha}\right),$$

where, $z, \alpha, \eta, \gamma, \theta > 0, \quad |\beta| \le 1$.

Merovci et al. [12] studied the transmuted generalized inverse Weibull (TGIW) distribution defined by CDF

$$G_{TGIW}(z) = e^{-\gamma(\lambda x)^\alpha}\left(1 + \beta - \beta e^{-\gamma(\lambda x)^\alpha}\right),$$

where, $z, \alpha, \lambda, \gamma > 0, \quad |\beta| \le 1$.

By using (1) and (2) in (3) and (4) we get the CDF and PDF of the TFWEx model, respectively. The key objectives of the present article is to propose a new lifetime model having a closed form of CDF and capable of modeling with increasing, unimodal or modified unimodal failure rates. The proposed model is very flexible and due to its increasing and modified unimodal failure rate, it can be useful for modeling the lifecycle of a machine's component in reliability engineering and lifecycle of cancer patients in biological studies. This article is organized as follows: Section 2,

contains definition and graphical display of the TFWEx model. Section 3, derives basic properties. Section 4, 5, 6 and 7 discusses the moment generating, characteristics, probability generating, and factorial moment generating functions of the model, respectively. Section 8, derives the maximum likelihood estimates and confidence bounds of the unknown parameters. Section 9, discusses orders statistics; Section 10, contains the analysis of real data sets using the proposed model. Finally, section 11, provides concluding comments.

2. TRANSMUTED FLEXIBLE WEIBULL EXTENSION DISTRIBUTION

A transmuted flexible Weibull extended random variable has the CDF given by

$$G(z; \theta, \lambda, \beta) = \left(1 - e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right\}, \quad z > 0,$$

(5)

here $\theta, \lambda > 0$, while $|\beta| \le 1$. Using $\beta = 0$, in (5), we get the CDF of FWEx distribution. While, using $\beta = \theta = 0$, and $\gamma = \ln(\lambda)$, in (5), we have the CDF of exponential distribution. Also, using $\theta = 0$, and $\gamma = \ln(\lambda)$, in (5), we get the CDF of transmuted exponential (TE) distribution.

The density associating to (5) is

$$g(z; \theta, \lambda, \beta) = \left(\lambda + \frac{\theta}{z^2}\right)e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}$$
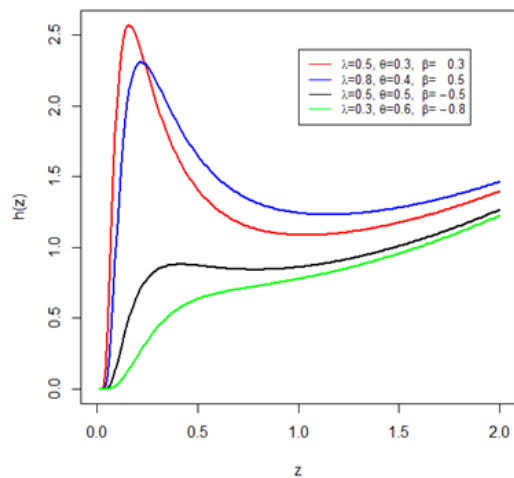$$\times \left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right\}.$$

The SF associating to (5) is

$$S(z; \theta, \lambda, \beta) = 1 - \left[\left(1 - e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right\}\right].$$

The HF corresponding to (5) is

$$h(z; \theta, \lambda, \beta) = \frac{\left(\lambda + \frac{\theta}{z^2}\right)e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right\}}{1 - \left[\left(1 - e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right\}\right]}.$$
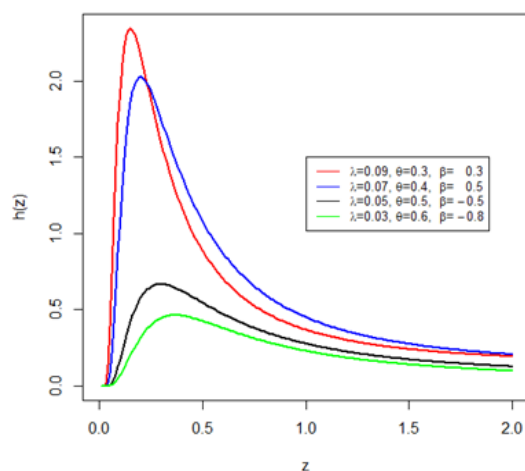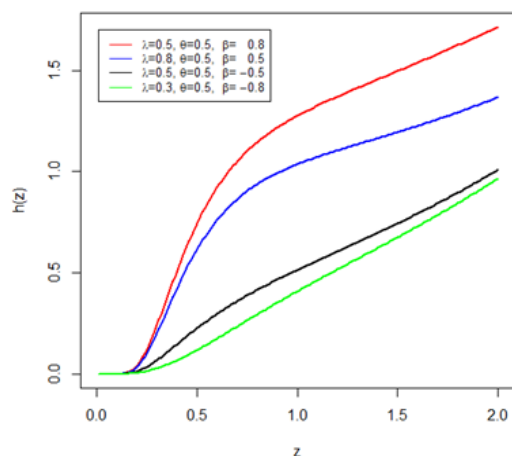
The figure 1, 2 & 3 shows the HF's of the TFWEx distribution for different parameter values.

Figure 1: HF of TFWEx distribution for different parameter values.

Figure 2: HF of TFWEx distribution, for different parameter values.

Figure 3: HF of TFWEx distribution, for different parameter values.

3. BASIC PROPERTIES

This part of the article provides the basic mathematical properties of TFWEx distribution.

3.1 Quantile and Median

The qth quantile denoted by z_q of the TFWEx model is obtained as

$$G\left(z_q\right) = q,$$

$$\left(1 - e^{-e^{\left(\lambda z_q - \frac{\theta}{z_q}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_q - \frac{\theta}{z_q}\right)}}\right\} = q.$$

On solving, we get

,

$$\lambda z_q^2 - \ln\left[-\ln\left\{\frac{(\beta-1) + \sqrt{(1-\beta)^2 - 4\beta(q-1)}}{2\beta}\right\}\right]z_q - \theta = 0.$$

$$(6)$$

Let

$$I = \ln\left[-\ln\left\{\frac{(\beta-1) + \sqrt{(1-\beta)^2 - 4\beta(q-1)}}{2\beta}\right\}\right],$$

So (6) becomes

$$\lambda z_q^2 - I z_q - \theta = 0.$$

Finally, we get

$$z_q = \frac{I + \sqrt{I^2 + 4\lambda\theta}}{2\lambda} \quad . \qquad (7)$$

Using $q = 0.50,$ in (7), we get the median of TFWEx distribution. Also, using $q = 0.25,$ and $q = 0.75,$ in (7), one may get, the 1st and 3rd quartile of TFWEx distribution, respectively.

3.2 Moments

Moments play a fundamental role in describing the shape and behaviour of a statistical model. In this subsection, we derive the rth moments of TFWEx distribution.

Theorem 1: If Z has TFWEx distribution, then

the rth moments of Z denoted by $\mu_r^{'}$ is given by

$$\mu_r^{/} = \sum_{i=o}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\theta)^j \Gamma(r-j-1)}{i!j!(\lambda)^{r-j}(i+1)^{r-2j-1}} \begin{bmatrix} (1-\beta)\left\{\left(\frac{(r-j)(r-j-1)}{(i+1)^2}+\lambda\theta\right)\right\} \\ +2^{i+1}\beta\left\{\left(\frac{(r-j)(r-j-1)}{(i+1)^2}+\lambda\theta\right)\right\} \end{bmatrix}.$$

Proof:

By definition, we have

$$\mu_r^{/} = \int_0^{\infty} z^r g(z)\,dz,$$

$$\mu_r^{/} = \int_0^{\infty} z^r\left(\lambda+\frac{\theta}{z^2}\right)e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\left\{1-\beta+2\beta e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}\right\}dz,$$

$$\mu_r^{/} = \lambda(1-\beta)\int_0^{\infty} z^r\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz$$

$$+\theta(1-\beta)\int_0^{\infty} z^{r-2}\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz$$

$$+2\lambda\beta\int_0^{\infty} z^r\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-2e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz$$

$$+2\theta\beta\int_0^{\infty} z^{r-2}\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-2e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz. \qquad (8)$$

Let

$$W_1 = \int_0^{\infty} z^r\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz,$$

$$W_2 = \int_0^{\infty} z^{r-2}\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz,$$

$$W_3 = \int_0^{\infty} z^r\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-2e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz,$$

And

$$W_4 = \int_0^{\infty} z^{r-2}\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-2e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz.$$

Therefore, (8) can be written as

$$\mu_r^{/} = \lambda(1-\beta)W_1 + \theta(1-\beta)W_2 + 2\lambda\beta W_3 + 2\theta\beta W_4. \qquad (9)$$

Consider,

$$W_1 = \int_0^{\infty} z^r\, e^{\left(\lambda z - \frac{\theta}{z}\right)}e^{-e^{\left(\lambda z - \frac{\theta}{z}\right)}}dz, \qquad (10)$$

From Taylor series, we have

$$e^{-tz} = \sum_{i=o}^{\infty}\frac{(-t)^i (z)^i}{i!}.$$

So,

$$e^{-e^{\left(\frac{\theta}{z}-\lambda z\right)}} = \sum_{i=o}^{\infty}\frac{(-1)^i\left(e^{\left(\lambda z - \frac{\theta}{z}\right)}\right)^i}{i!}.$$

Using above transformation in (10), we get

$$W_1 = \sum_{i=o}^{\infty}\frac{(-1)^i}{i!}\int_0^{\infty} z^r\left(e^{\left(\lambda z - \frac{\theta}{z}\right)}\right)^{i+1}dz,$$

$$W_1 = \sum_{i=o}^{\infty}\frac{(-1)^i}{i!}\int_0^{\infty} z^r e^{-\frac{\theta(i+1)}{z}}\, e^{\lambda(i+1)z}\,dz,$$

$$W_1 = \sum_{i=o}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\theta)^j(i+1)^j}{i!j!}\int_0^{\infty} z^{r-j}\, e^{\lambda(i+1)z}\,dz. \qquad (11)$$

Using the definition of gamma function

$$\Gamma(t) = z^t\int_0^{\infty} e^{tz}t^{z-1}\,dt.$$

Finally, we get

$$\sum_{i=o}^{\infty}\sum_{j=}^{\infty}\frac{(-1)^{i\ j}(\ )^j \Gamma(r-j+1)}{i!j!(\ )^{(r-j+1)}(i\ 1)^{(r-2j+1)}}. \qquad (12)$$

Similarly,

$$W_2 = \sum_{i=o}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\theta)^j \Gamma(r-j-1)}{i!j!(\lambda)^{r-j-1}(i+1)^{r-2j-1}}. \qquad (13)$$

$$W_3 = \sum_{i=o}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\theta)^j (2)^i \Gamma(r-j+1)}{i!j!(\lambda)^{(r-j+1)}(i+1)^{(r-2j+1)}}. \qquad (14)$$

And

$$W_4 = \sum_{i=o}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(\theta)^j (2)^i \Gamma(r-j-1)}{i!j!(\lambda)^{r-j-1}(i+1)^{r-2j-1}}. \qquad (15)$$

Using (12), (13), (14) and (15) in (9), we get the proof of thereon 1.

3.3 Generation of Random Numbers

The random number as *z* of the TFWEx distribution can be obtained as

$$G(z) = R, \qquad \text{Where } R \sim U(0,1)$$

$$\left(1-e^{-e^{\left(\lambda z-\frac{\theta}{z}\right)}}\right)\left\{1+\beta e^{-e^{\left(\lambda z-\frac{\theta}{z}\right)}}\right\}=R.$$

On solving, we get

$$\lambda z^2-\ln\left[-\ln\left\{\frac{(\beta-1)+\sqrt{(1-\beta)^2-4\beta(R-1)}}{2\beta}\right\}\right]z-\theta=0.$$

$$(16)$$

Let

$$I=\ln\left[-\ln\left\{\frac{(\hat{a}-1)+\sqrt{(1-\hat{a})^2-4\hat{a}(R-1)}}{\hat{a}}\right\}\right],$$

so (16) become

$$\lambda z^2-Iz-\theta=0.$$

Finally, we get

$$z=\frac{I+\sqrt{I^2+4\lambda\theta}}{2\lambda}.\qquad(17)$$

It is observed that the expression for generation of random numbers from TFWEx distribution has a closed form solution, so one can use (7) to generate random numbers from TFWEx distribution.

4.   MOMENT GENERATING FUNCTION

The moment generating function (MGF) is a prominent approach to generate moments of a statistical model. If Z has TFWEx distribution, then its MGF denoted by $M_z(t)$ is derived as

$$M_z(t)=E(e^{tz}),$$
$$M_z(t)=\int_0^\infty e^{tz}g(z)dz,$$

$$M_z(t)=\sum_{r=0}^\infty\frac{\log^r(t)}{r!}\int_0^\infty z^r g(z)dz.\qquad(18)$$

Using the result of theorem 1, in (18), one may easily derive the MGF of TFWEx distribution.

5.   CHARACTERISTIC FUNCTION

The moment generating function is frequently used to generate moments a distribution. But, the MGF does not exist for all distributions. Therefore, alternatively one can use character-istic function (CF) to calculate moments of a statistical distribution. If Z has TFWEx distribution, then its CF represented by $\phi_z(t)$ is obtained as

$$\phi_z(t)=E(e^{itz}),$$
$$\phi_z(t)=\int_0^\infty e^{itz}g(z)dz,$$

$$\phi_z(t)=\sum_{r=0}^\infty\frac{\log^r(it)}{r!}\int_0^\infty z^r g(z)dz.\qquad(19)$$

Using the result derived in theorem 1, in (19), one can easily obtain the CF of TFWEx distribution.

6.   PROBABILITY GENERATING FUNCTION

The probability generating function (PGF) of TFWEx random variable can be derived as

$$G(\gamma)=E(\gamma^z)$$

$$G(\gamma)=\int_0^\infty e^{z\ln(\gamma)}g(z)dz,$$

$$G(\gamma)=\sum_{r=0}^\infty\frac{\log^r(\gamma)}{r!}\int_0^\infty z^r g(z)dz.\qquad(20)$$

One may get the complete proof of the PGF of TFWEx distribution by using the result of theorem 1, in (20).

7.   FACTORIAL MOMENT GENERATING FUNCTION

The factorial moment generating function (FMGF) of TFWEx random variable can be obtained as

By definition

$$H_0(\delta)=E\left\{(1+\delta)^z\right\},$$

$$H_0(\delta)=\int_0^\infty e^{z\ln(1+\delta)}g(z)dz,$$

$$H_0(\delta)=\sum_{r=0}^\infty\frac{\log^r(1+\delta)}{r!}\int_0^\infty z^r g(z)dz.\qquad(21)$$

One may obtain the complete proof of the FMGF of TFWEx distribution by using the result of the-

orem 1, in (21).

8. ESTIMATION

In this section, we use maximum likelihood estimation (MLE) procedure for estimating the unknown parameters of the TFWEx distribution, and deriving their asymptotic confidence bounds.

8.1 Maximum likelihood estimation

Let a sample say $Z_1, Z_2, \ldots Z_k$ obtained from TFWEx with parameters (θ, λ, β) then the likelihood function of this sample is

$$L = \prod_{j=0}^{k} g\left(z_j; \theta, \lambda, \beta\right),$$

$$L = \prod_{j=0}^{k}\left[\left(\lambda + \frac{\theta}{z_j^2}\right) e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)} e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right\}\right],$$

the log-likelihood function is as

$$\ln L = \sum_{j=0}^{k} \ln\left(\lambda + \frac{\theta}{z_j^2}\right) + \lambda \sum_{j=0}^{k} z_j - \theta \sum_{j=0}^{k} \frac{1}{z_j} - \sum_{j=0}^{k} e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}$$

$$+ \sum_{j=0}^{k} \ln\left(1 - \beta + 2\beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right). \quad (22)$$

Taking the partial derivatives of the result in (22) on parameter and equating to zero, we have

$$\frac{\partial \ln L}{\partial \theta} = \sum_{j=0}^{k} \frac{1}{\left(\lambda z_j^2 + \theta\right)} - \sum_{j=0}^{k} \frac{1}{z_j} + \sum_{j=0}^{k} \frac{e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}{z_j}$$

$$+ 2\beta \sum_{j=0}^{k} \frac{e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)} e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}}{\left(1 - \beta + 2\beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right) z_j} = 0. \quad (23)$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{j=0}^{k} \frac{z_j^2}{\left(\lambda z_j^2 + \theta\right)} + \sum_{j=0}^{k} z_j - \sum_{j=0}^{k} z_j e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}$$

$$- 2\beta \sum_{j=0}^{k} \frac{z_j e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)} e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}}{\left(1 - \beta + 2\beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right)} = 0. \quad (24)$$

$$\frac{\partial \ln L}{\partial \beta} = \sum_{j=0}^{k} \frac{2 e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}} - 1}{\left(1 - \beta + 2\beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right) z_j} = 0. \quad (25)$$

It is clear that the expressions provided in (23)-(25) does not have a closed forms solution; therefore, the estimates of the model parameters can be obtained numerically by iterating procedure such as newton Raphson method. We used "SANN" algorithm in R language to get numerical estimates of the unknown parameters of the proposed model.

8.2 Asymptotic confidence bounds

As it is observed that expressions provided in (23)-(25) are not in closed forms. Therefore, it is quite difficult to obtain the exact distribution of the MLE's. So, it is better to derive the asymptotic confidence intervals of the unknown parameters. The most frequently used approach is to assume that the MLE's $\left(\hat{\theta}, \hat{\lambda}, \hat{\beta}\right)$ are distributed approximately normal with mean (θ, λ, β) and covariance matrix All the second order derivatives for the density of TFWEx distribution exist. Thus we have

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda} \\ \hat{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \theta \\ \lambda \\ \beta \end{pmatrix}, \Sigma\right],$$

with

$$\Sigma = -E\begin{bmatrix} V_{\theta\theta} & V_{\theta\lambda} & V_{\theta\beta} \\ V_{\lambda\theta} & V_{\lambda\lambda} & V_{\lambda\beta} \\ V_{\beta\theta} & V_{\beta\lambda} & V_{\beta\beta} \end{bmatrix}^{-1}.$$

Where

$$V_{\theta\theta} = \frac{\partial^2 \ln L}{\partial \theta^2} \quad V_{\theta\lambda} = \frac{\partial^2 \ln L}{\partial \theta \partial \lambda} \quad V_{\theta\beta} = \frac{\partial^2 \ln L}{\partial \theta \partial \beta}$$

$$V_{\lambda\theta} = \frac{\partial^2 \ln L}{\partial \lambda \partial \theta} \quad V_{\lambda\lambda} = \frac{\partial^2 \ln L}{\partial \lambda^2} \quad V_{\lambda\beta} = \frac{\partial^2 \ln L}{\partial \lambda \partial \beta}$$

$$V_{\beta\theta} = \frac{\partial^2 \ln L}{\partial \beta \partial \theta} \quad V_{\beta\lambda} = \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \quad V_{\beta\beta} = \frac{\partial^2 \ln L}{\partial \beta^2}$$

Since contains unknown parameters, to get estimate of, so we replace the unknown parameters by their corresponding MLE's, given by

$$\hat{\Sigma} = \begin{bmatrix} \hat{V}_{\theta\theta} & \hat{V}_{\theta\lambda} & \hat{V}_{\theta\beta} \\ \hat{V}_{\lambda\theta} & \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\beta} \\ \hat{V}_{\beta\theta} & \hat{V}_{\beta\lambda} & \hat{V}_{\beta\beta} \end{bmatrix}^{-1}. \quad (26)$$

Using (25), approximately $100(1-\alpha)\%$ % confidence intervals for θ, λ and β can be determined respectively, as

$$\hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\theta\theta}} , \qquad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\lambda\lambda}} , \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{\beta\beta}} .$$

Here, $Z_{\frac{\alpha}{2}}$ represents the upper $\left(\frac{\alpha}{2}\right)^{th}$ percentile of the standard normal (SN) distribution.

9. ORDER STATISTICS

Let $Z_1, Z_2 \ldots Z_k$ are independently and identically distributed (i.i.d) ordered random variables taken from TFWEx with parameters (θ, λ, β) in such a way that $Z_{(1:k)} \leq \ldots \leq Z_{(k:k)}$. Then, the density of $Z(i:k)$, $i=1, 2, 3, \ldots, k$ is

$$g_{i:k}(z) = \frac{1}{Beta(i, n-i+1)} g(z, \Phi)$$
$$\times \left[G(z, \Phi) \right]^{i-1} \left[1 - G(z, \Phi) \right]^{k-i} . \qquad (27)$$

Where $\Phi = (\theta, \lambda, \beta)$. Also, the joint density of $((z_{i:k}), (z_{k:k}))$ is

$$g_{i:j:k}(z_i, z_j) = C \left[G(z_i) \right]^{i-1} \left[G(z_j) - G(z_i) \right]^{j-i-1}$$
$$\times \left[1 - G(z_j) \right]^{k-j} g(z_i) g(z_j),$$

Where

$$C = \frac{k!}{(i-1)!(j-i-1)!(k-j)!} .$$

We derive the expressions, for the kth order statistics as $Z_{(k)} = \max(Z_1, Z_2, \ldots, Z_k)$, 1st order statistics as $Z_{(1)} = \min(Z_1, Z_2, \ldots, Z_k)$, and for the median order statistics as Z_{m+1}, if $k = 2m+1$.

9.1 Distribution of Maximum, Minimum and Median Order Statistics

Let a random sample $Z_1, Z_2 \ldots Z_k$ of size k se

lected from TFWExD(θ, λ, β) with CDF given in (5). Then, the density of the maximum, minimum and median order statistics is derived in (32), (33) and (34) respectively.

$$g_{k:k}(z) = kg(z) \left[G(z) \right]^{k-1} ,$$

$$g_{k:k}(z) = k \left(\lambda + \frac{\theta}{z_k^2} \right) e^{\left(\lambda z_k - \frac{\theta}{z_k} \right)} e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k} \right)}} \left\{ 1 - \beta + 2\beta e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k} \right)}} \right\}$$
$$\times \left[\left(1 - e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k} \right)}} \right) \left\{ 1 + \beta e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k} \right)}} \right\} \right]^{k-1} . \qquad (28)$$

And,

$$g_{1:k}(z) = k g(z) \left[1 - G(z) \right]^{k-1} .$$

$$g_{1:k}(z) = k \left(\lambda + \frac{\theta}{z_1^2} \right) e^{\left(\lambda z_1 - \frac{\theta}{z_1} \right)} e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1} \right)}} \left\{ 1 - \beta + 2\beta e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1} \right)}} \right\}$$
$$\times \left[1 - \left(1 - e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1} \right)}} \right) \left\{ 1 + \beta e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1} \right)}} \right\} \right]^{k-1} . \qquad (29)$$

Also, the PDF of median order statistics is

$$g_{m+1:k}(\tilde{z}) = \frac{(2m+1)!}{m!m!} g(\tilde{z}) \left\{ G(\tilde{z}) \right\}^m \left\{ 1 - G(\tilde{z}) \right\}^m ,$$

$$g_{m+1:k}(\tilde{z}) = \frac{(2m+1)!}{m!m!} \left(\lambda + \frac{\theta}{z^2} \right) e^{\left(\lambda z - \frac{\theta}{z} \right)} e^{-e^{\left(\lambda z - \frac{\theta}{z} \right)}}$$
$$\times \left\{ 1 - \beta + 2\beta e^{-e^{\left(\lambda z - \frac{\theta}{z} \right)}} \right\}$$
$$\times \left\{ 1 - \left(1 - e^{-e^{\left(\lambda z - \frac{\theta}{z} \right)}} \right) \left\{ 1 + \beta e^{-e^{\left(\lambda z - \frac{\theta}{z} \right)}} \right\} \right\}^m$$
$$\times \left\{ \left(1 - e^{-e^{\left(\lambda z - \frac{\theta}{z} \right)}} \right) \left\{ 1 + \beta e^{-e^{\left(\lambda z - \frac{\theta}{z} \right)}} \right\} \right\}^m . \qquad (30)$$

9.2 The joint density of i-th and j-th Order Statistics

The joint density of ith and jth order statistics from TFWEx distribution is

$$g_{i:j:k}(z_i, z_j) = \frac{k!}{(i-1)!(j-i-1)!(k-j)!} \left[G(z_i) \right]^{i-1}$$
$$\times \left[G(z_j) - G(z_i) \right]^{j-i-1} \left[1 - G(z_j) \right]^{k-j}$$
$$\times g(z_i) g(z_j).$$

Let $\qquad \overline{(i-1)!(j-i-1)!(k-j)!}$

Thus, the joint density of ith and jth order statis-

tics is

$$g_{i:j:k}(z_i, z_j) = C\left[\left(1 - e^{-e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)}}\right\}\right]^{i-1}$$

$$\times \left[\left(1 - e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right\} - \left(1 - e^{-e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)}}\right\}\right]^{j-i-1}$$

$$\times \left[1 - \left(1 - e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right\}\right]^{k-j}$$

$$\times \left(\lambda + \frac{\theta}{z_i^2}\right) e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)} e^{-e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)}}\left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z_i - \frac{\theta}{z_i}\right)}}\right\}$$

$$\times \left(\lambda + \frac{\theta}{z_j^2}\right) e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)} e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z_j - \frac{\theta}{z_j}\right)}}\right\}.$$

For special case: let $j=k$ and $i=1$, we get the joint density of maximum and minimum order statistics

$$g_{1:k:k}(z_1, z_k) = k(k-1)\left[G(z_k) - G(z_1)\right]^{k-2} g(z_1)g(z_k),$$

so, we get

$$g_{1:k:k}(z_1, z_k) = k(k-1)\left[\left(1 - e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k}\right)}}\right\} - \left(1 - e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1}\right)}}\right)\left\{1 + \beta e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1}\right)}}\right\}\right]^{k-2}$$

$$\times \left(\lambda + \frac{\theta}{z_1^2}\right) e^{\left(\lambda z_1 - \frac{\theta}{z_1}\right)} e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1}\right)}}\left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z_1 - \frac{\theta}{z_1}\right)}}\right\}$$

$$\times \left(\lambda + \frac{\theta}{z_k^2}\right) e^{\left(\lambda z_k - \frac{\theta}{z_k}\right)} e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k}\right)}}\left\{1 - \beta + 2\beta e^{-e^{\left(\lambda z_k - \frac{\theta}{z_k}\right)}}\right\}.$$

10. APPLICATION

In this section, we analyze four data sets in order to prove the practicality of the transmuted flexible Weibull extension distribution. We applied the TFWEx distribution to four well-known real data sets, and the result of its goodness of fit is compared with flexible Weibull extension and inverse flexible Weibull extension (IFWEx) distribution proposed by El-Gohary et al [8]. We considered Akaike's Information Criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike's Information Criterion (CAIC), Hannan-Quinn information criterion (HQIC), Anderson–Darling (AD) test statistic, Cramer-von-Misses (C.M) test statistic and Kolmogorov–Smirnov (K-S) test statistic as investigative measures. On behalf of these investigative tools, it observed that TFWEx distribution provides best fit than the FWEx and IFWEx distributions.

Example: 1

The first data set is obtained from Nicholas and Padgett [14], consists of 100 observations on breaking stress of carbon fibers (in Gba). The data are as: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, and 3.65. The TFWEx distribution along with FWEx and IFWEx distribution is applied to this data and the final result is summarized in table 1 and 1.1.

Table 1: Goodness of fit results for TFWEx, FWEx and IFWEx.

Dist.	MLEs	A.D	C.M	K-S
TFWEx	$\hat{}$ =**0.41** $\hat{\lambda}$ =**2.34** $\hat{\beta}$ =**-0.81**	**0.50**	**0.08**	**0.06**
FWEx	$\hat{\theta}$ =0.43, $\hat{\lambda}$ = 3.69	0.64	0.11	0.07
IFWEx	$\hat{\theta}$ = 1.13, $\hat{\lambda}$ =0.49	3.82	0.64	0.25

Table 1.1: Goodness of fit results for TFWEx, FWEx and IFWEx.

Dist.	AIC	BIC	CAIC	HQIC
TFWEx	**289.9**	**297.7**	**290.1**	**293.0**
FWEx	290.5	**295.7**	290.6	292.6
IFWEx	360.1	365.3	360.2	362.2

Example: 2

The second data set is obtained from Khan and Jan [11] represents the times of failure for a sample of thirty devices selected from eld-tracking study of a larger system. The times are 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45 and 2.66. The TFWEx distribution along with FWEx and IFWEx distribution is applied to this data and the final result is summarized in table 2 and 2.1.

Table 2: Goodness of fit results for TFWEx, FWEx and IFWEx

Dist.	MLEs	A.D	C.M	K-S
TFWEx	$\hat{\theta} = \mathbf{0.09},\ \hat{\lambda} = \mathbf{0.35}$ $\hat{\beta} = \mathbf{-0.9}$	**1.52**	**0.22**	**0.28**
FWEx	$\hat{\theta} = 0.15,\ \hat{\lambda} = 0.32$	2.04	0.32	0.39
IFWEx	$\hat{\theta} = 0.03,\ \hat{\lambda} = 0.62$	1.60	0.24	0.29

Table 2.1: Goodness of fit results for TFWEx, FWEx and IFWEx

Dist.	AIC	BIC	CAIC	HQIC
TFWEx	**97.4**	**101.6**	**98.39**	**98.8**
FWEx	111.3	114.1	111.7	112.2
IFWEx	106.3	109.1	106.7	107.2

Example: 3

The third data set is obtained from Tahir et al. [19] represents failure times of 84 Aircraft Windshield. The times are 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376 and 4.663. The TFWEx distribution along with FWEx and IFWEx distribution is applied to this data and the final result is summa-

rized in table 3 and 3.1.

Table 3: Goodness of fit results for TFWEx, FWEx and IFWEx.

Dist.	MLE	A.D	C.M	K-S
TFWEx	$\hat{\theta} = \mathbf{0.31},\ \hat{\lambda} = \mathbf{0.83}$ $\hat{\beta} = \mathbf{-0.93}$	**3.21**	**0.47**	**0.23**
FWEx	$\hat{\theta} = 0.30,\ \hat{\lambda} = 1.39$	5.59	0.90	0.31
IFWEx	$\hat{\theta} = 0.06,\ \hat{\lambda} = 0.49$	1.82	0.22	0.48

Table 3.1: Goodness of fit results for TFWEx, FWEx and IFWEx.

Dist.	AIC	BIC	CAIC	HQIC
TFWEx	**316.5**	**323.9**	**316.8**	**319.5**
FWEx	355.6	360.5	355.7	357.6
IFWEx	378.4	383.3	378.6	380.4

Example: 4

The fourth data set is obtained from Tahir et al. [19] represents Failure times of 63 Aircraft Windshield. The times are 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543 and 5.140. The TFWEx distribution along with FWEx and IFWEx distribution is applied to this data and the final result is summarized in table 4 and 4.1.

Table 4: Goodness of fit results for TFWEx, FWEx and IFWEx.

Dist.	MLE	A.D	C.M	K-S
TFWEx	$\hat{\theta} = \mathbf{0.31},\ \hat{\lambda} = \mathbf{0.32}$ $\hat{\beta} = \mathbf{-\ 0.92}$	**1.51**	**0.26**	**0.23**
FWEx	$\hat{\theta} = 0.29,\ \hat{\lambda} = 0.6$	2.92	0.52	0.32
IFWEx	$\hat{\theta} = 0.082,\ \hat{\lambda} = 0.56$	1.26	0.20	0.37

Table 4.1: Goodness of fit results for TFWEx, FWEx and IFWEx.

Dist.	AIC	BIC	CAIC	HQIC
TFWEx	**221.05**	**227.47**	**221.45**	**223.57**
FWEx	242.70	246.99	242.90	244.39
IFWEx	245.23	249.51	245.43	246.91

11. CONCLUSION

A three parameters transmuted version of the flexible Weibull extension distribution entitled as Transmuted Flexible Weibull Extension distribution by adopting the quadratic rank transmutation map is studied. The new model is capable of modeling data with increasing, unimodal or modified unimodal failure rates. Statistical properties of the proposed distribution with estimation of parameters by maximum likelihood procedure are discussed. To show practical workability of the suggested model four real data sets are analyzed. We are hopeful that the proposed model will attract a widespread applications in survival and reliability disciplines.

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