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Generalization of Hadamard Matrices

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ABSTRACT

Hadamard's matrices are used widely at the forward links of communication channels to mix the information on connecting to and at the backward links of these channels to sift through this information is transmitted to reach the receivers this information in correct form, specially in the pilot channels, the Sync channels, the traffic channel and so much applications in the fields; Modern communication and telecommunication systems, signal processing, optical multiplexing, error correction coding, and design and analysis of statistics.

This research is useful to generate new sets of orthogonal matrices by generalization Hadamard matrices,with getting bigger lengths and bigger minimum distance by using binary representation of the matrices that assists to increase secrecy of these information, increase the possibility of correcting mistakes resulting in the channels of communication, giving idea to construct new coders and decoders by mod p with more complexity for using these matrices and derivation new orthogonal codes or sequences.

Keyword: Hadamard matrix, Binary vector, Coefficient of Correlation, Walsh's Sequences, Orthogonal sequences, Kronecker product, Code

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Introduction

Hadamard matrices seem such simple matrix structures: they are square, have entries +1 or −1 and have orthogonal row vectors and orthogonal column vectors.

A Hadamard matrix is invented by Sylvester (1867), 26 years before Hadamard (1893) considered them. The $n \times n$ Hadamard matrix H_n must have $n(n-1)/2$ of "-1.s" and $n(n+1)/2$ of "1.s".

A Hadamard matrix of order *n* is a solution to Hadamard's maximum determinant problem, i.e., has the maximum possible determinant (in absolute value) of any $n \times n$ complex matrix with elements $|a_{ij}| \le 1$ (Brenner and Cummings 1972), namely $n^{n/2}$. An equivalent definition of the Hadamard matrices is given by $H_n H_n^T = nI_n$, where I_n is the $n \times n$ identity matrix.

Indeed, using the matrix of order 1, *Sylvester (1867)* proved " there is an Hadamard matrix of order 2^t for all non-negative t .

The matrices of order 2^t constructed using Sylvester's construction are usually referred to as Sylvester-Hadamard matrices. The Sylvester-Hadamard matrices are associated with discrete orthogonal functions called Walsh functions. Hadamard (1893) gave examples for a few small orders.

Hadamard (1893) remarked that a necessary condition for a Hadamard matrix to exist is that n =1, 2, or a positive multiple of 4 (Brenner and Cummings 1972). Paley's theorem guarantees that there always exists a Hadamard matrix H_n when n is divisible by 4 and of the form $2^e(p^m+1)$ for some positive integer *m*, nonnegative integer *e*, and *p* an odd prime . In such cases, the matrices can be constructed using a Paley construction.

Yet they have been actively studied for over 138 years and still have more secrets to be discovered.

Research method and Material

***Definition* 1.** The complement of the binary vector $X = (x_1, x_2, ..., x_n),\ x_i \in F_2\{0,1\}$ is the vector $\overline{X} = (\overline{x_1}, \overline{x_2}, ..., \overline{x_n})$, where:

$$\overline{x_i} = \begin{cases} 1 & if \quad x_i = 0 \\ 0 & if \quad x_i = 1 \end{cases}. \qquad [1,2]$$

***Definition* 2.** Suppose $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0, y_1, ..., y_{n-1})$ are binary vectors of length n on *GF*(2)={0,1}. The coefficient of correlations function of *x* and *y*, denoted by $R_{x,y}$, is:

$$R_{x,y} = \sum_{i=0}^{n-1} (-1)^{x_i + y_i} \qquad (1)$$

Where $x_i + y_i$ is computed *mod* 2. It is equal to the number of agreements components minus the number of disagreements corresponding to components or if $x_i, y_i \in \{1, -1\}$ (usually, replacing in binary vectors *x* and *y* each "1" by "-1" and each "0" by "1") then

$$R_{x,y} = \sum_{i=0}^{n-1} x_i y_i \text{ , [1,2],[13-18]} \qquad (2)$$

***Definition* 3**. Suppose $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0, y_1, ..., y_{n-1})$ are binary vectors of length *n* on *GF*(2)={0,1}, or components belong to {1, -1}, is said strictly orthogonal or briefly orthogonal if $R_{x,y} = 0$, (Usually, said orthogonal if $R_{x,y} \in \{-1, 0.1\}$). [1,2],[13-18]

***Definition* 4.** Suppose *G* is a set of binary vectors of length *n*:

$$G = \{X; X = (x_0, x_1, ..., x_{n-1}), x_i \in F_2 = \{0,1\}, i = 0,1,...,n-1\}$$

Let's 1^*= -1 and 0^* =1, The set *G* is said to be orthogonal if the following two conditions are Satisfied:

$$1.\ \forall X \in G,\ \sum_{i=0}^{n-1} x_i^* = 0, or \left|R_{x,0}\right| = 0. \qquad (3)$$

$$2.\ \forall X, Y \in G (X \ne Y),\ \sum_{i=0}^{n-1} x_i^* y_i^* = 0\ or \left|R_{x,y}\right| = 0. \qquad (4)$$

That is, the absolute value of "the number of agreements minus the number of disagreements " is equal to zero. [1,2], [13-18]

***Definition* 5**. The matrix $A=[a_{ij}]$ is called Hadamard matrix if it is a square matrix and each entry is equals 1 or ”-“ (where - denotes 1) with the property that if the size of *A* is *h* then $A^T A = AA^T = hI_h$, in the decimal counting system, and the distinct rows vectors are mutually orthogonal. [20-27]

***Definition* 6**. If all entries of the first row and the first column in the Hadamard matrix are equal to one then the matrix is called standard Hadamard matrix. The smallest examples are:

$$H_{2^0}=[1],\quad H_{2^1}=\begin{bmatrix}1&1\\1&-\end{bmatrix},\quad H_{2^2}=\begin{bmatrix}1&1&1&1\\1&-&1&-\\1&1&-&-\\1&-&-&1\end{bmatrix},\quad H_{2^3}=\left[\begin{array}{cccc|cccc}1&1&1&1&1&1&1&1\\1&-&1&-&1&-&1&-\\1&1&-&-&1&1&-&-\\1&-&-&1&1&-&-&1\\\hline 1&1&1&1&-&-&-&-\\1&-&1&-&-&1&-&1\\1&1&-&-&-&-&1&1\\1&-&-&1&-&1&1&-\end{array}\right]$$

Such matrices were first invented by Sylvester (1867) who observed that if *H* is an Hadamard matrix, then:

$$H=\begin{bmatrix}H&H\\H&-H\end{bmatrix} \qquad (5)$$

Is also an Hadamard matrix. [3-12]

***Lema* 1**. (Sylvester (1867)): There is an Hadamard matrix of order 2^t for all nonnegative *t*.

The matrices of order 2^t constructed using Sylvester’s construction are usually Referred to as Sylvester-Hadamard matrices. [3-10]

Hadamard (1983) gave examples for a few small orders. An example of order 12 is as follows:

$$H_{12}=\left[\begin{array}{ccc|ccc|ccc|ccc}1&1&1&-&1&1&-&1&1&-&1&1\\1&1&1&1&-&1&1&-&1&1&-&1\\1&1&1&1&1&-&1&1&-&1&1&-\\\hline 1&-&-&1&1&1&-&1&1&1&-&-\\-&1&-&1&1&1&1&-&1&-&1&-\\-&-&1&1&1&1&1&1&-&-&-&1\\\hline 1&-&-&1&-&-&1&1&1&-&1&1\\-&1&-&-&1&-&1&1&1&1&-&1\\-&-&1&-&-&1&1&1&1&1&1&-\\\hline 1&-&-&-&1&1&1&-&-&1&1&1\\-&1&-&1&-&1&-&1&-&1&1&1\\-&-&1&1&1&-&-&-&1&1&1&1\end{array}\right]$$

Some basic properties of Hadamard matrices are given by following theorem:

***Theorem* 2**: Let H_h be an Hadamard matrix of order *h* then:

1. $H_h H_h^t = hI_h$, where I_h is the identity matrix of order *h*;
2. $|\det H| = h^{\frac{1}{2}h}$;
3. $H_h H_h^t = H_h^t H_h$;
4. Hadamard matrices may be changed into other Hadamard matrices by permuting

rows and columns and by multiplying rows and columns by −1.

Matrices which can be obtained from one another by these methods are referred to as

H-equivalent (not all Hadamard matrices of the same order are *H*-equivalent).

5. Every Hadamard matrix is *H*-equivalent to an Hadamard matrix which has every element of its first row and column equal +1 – matrices of this latter form are called normalized.
6. If H_{4n} is a normalized Hadamard matrix of order 4*n*, then every row (column), except the first, has 2*n* minus ones and 2*n* plus ones in each row (column).
7. The order of an Hadamad matrix is 1, 2, 4*n*, where *n* is a positive integer.

(Sylvester) Let H_1 and H_2 be two Hadamard matrices of orders h_1 and h_2, then the

Kronecker product of H_1 and H_2 is an Hadamard matrix of order $h_1 h_2$.

8. Standard Hadamard matrix H_h have orthogonal rows (columns) vectors and each row (column) except the first row and first column contains h/2 of "1.s" and h/2 of "-1" and h/2 of disagreements and h/2 of agreements h/4 of agreements are "1.s" and h/4 of agreements are "-1".

The first unsolved case is order 668. The previous smallest unsolved case, 428, was found in 2004 by Kharaghani and Tayfeh-Rezaie Kharaghani and Tayfeh-Rezaie (2004).[3-12]

Results and Discussion

Suppose *p* is a prime number larger than 2 and the matrices:

i. $Z_{2^0}=[1]$, then: $Z_{2^0}^t=[1]$, $Z_{2^0}.Z_{2^0}^t=[1]$ and $\left|Z_{2^0}\right|=1$, $Z_{2^0}^{-1}=[1]$

ii. $Z_{2^1}=\begin{bmatrix}1 & 1\\ 1 & p-1\end{bmatrix}$, then:

$*\, Z_{2^1}=\begin{bmatrix}Z_{2^0} & Z_{2^0}\\ Z_{2^0} & \overline{Z}_{2^0}\end{bmatrix}$, where "$\overline{Z}_{2^0}=(p-1)Z_{2^0}$ & $(p-1)^2=1$" or is the same Z_{2^0} after replacing, in it, each "1" by "(*p*-1)" and, in the same time, each "(*p*-1)" by "1". Z_{2^1} Symmetric and $Z_{2^1}^t=Z_{2^1}$.

$$Z_{2^1}.Z_{2^1}^t=\begin{bmatrix}2 & p\\ p & 2+p(p-2)\end{bmatrix}=2^1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & p\\ p & p(p-2)\end{bmatrix}$$

$$=2^1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2^{1-1}p\\ 2^{1-1}p & 2^{1-1}p(p-2)\end{bmatrix}$$

$$*\, Z_{2^1}.Z_{2^1}^t=Z_{2^1}^t.Z_{2^1}=2^1 I_{2^1}+\begin{bmatrix}0 & 2^{1-1}p\\ 2^{1-1}p & 2^{1-1}p(p-2)\end{bmatrix}$$

$$*\left|Z_{2^1}\right|=(p-2)^{2^1-1},\ Z_{2^1}^{-1}=\frac{1}{p-2}\begin{bmatrix}p-1 & -1\\ -1 & 1\end{bmatrix}=\frac{1}{-(p-2)^{2^1-1}}\begin{bmatrix}-(p-1) & 1\\ 1 & -1\end{bmatrix}$$

$$Z_{2^1}^{-1} = \frac{1}{-(P-2)^{2^1-1}}\begin{bmatrix} \overline{\overline{H}}_{2^0} & 1.H_{2^0} \\ 1.H_{2^0} & 1.\overline{H}_{2^0} \end{bmatrix}$$

Where the $\overline{\overline{H}}_{2^0}$ is the same H_{2^0} after replacing the entry in the first row and first column by "- (p-1)" .

iii. * $Z_{2^2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & p-1 & 1 & p-1 \\ 1 & 1 & p-1 & p-1 \\ 1 & p-1 & p-1 & 1 \end{bmatrix}$, then:

* $Z_{2^2} = \begin{bmatrix} Z_{2^1} & Z_{2^1} \\ Z_{2^1} & \overline{Z}_{2^1} \end{bmatrix}$, where "$\overline{Z}_{2^1} = (p-1)Z_{2^1}$ & $(p-1)^2 = 1$" or is the same Z_{2^1} after replacing, in it, each "1" by "(p-1)" and, in the same time, each "(p-1)" by "1".

$$* \; Z_{2^2}^t = Z_{2^2} \,, \; Z_{2^2}.Z_{2^2}^t = \begin{bmatrix} 4 & 2p & 2p & 2p \\ 2p & 4+2p(p-2) & p^2 & p^2 \\ 2p & p^2 & 4+2p(p-2) & p^2 \\ 2p & p^2 & p^2 & 4+2p(p-2) \end{bmatrix}$$

$$* \; Z_{2^2}^t = Z_{2^2} \,, \; Z_{2^2}.Z_{2^2}^t = 2^2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2p & 2p & 2p \\ 2p & 2p(p-2) & p^2 & p^2 \\ 2p & p^2 & 2p(p-2) & p^2 \\ 2p & p^2 & p^2 & 2p(p-2) \end{bmatrix}$$

$$* \; Z_{2^2}^t = Z_{2^2} \,, \; Z_{2^2}.Z_{2^2}^t = 2^2 I_{2^2} + \left[\begin{array}{cc|cc} 0 & 2p & 2p & 2p \\ 2p & 2p(p-2) & p^2 & p^2 \\ \hline 2p & p^2 & 2p(p-2) & p^2 \\ 2p & p^2 & p^2 & 2p(p-2) \end{array}\right]$$

$$* \, Z_{2^2}.Z_{2^2}^t = Z_{2^2}^t.Z_{2^2} = 2^2 I_4 + \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2^{2-1}p \\ 2^{2-1}p & 2^{2-1}p(p-2) \end{bmatrix}, \; B = \begin{bmatrix} 2^{2-1}p & 2^{2-1}p \\ 2^{2-2}p^2 & 2^{2-2}p^2 \end{bmatrix} = \begin{bmatrix} 2p & 2p \\ p^2 & p^2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2p(p-2) & p^2 \\ p^2 & 2p(p-2) \end{bmatrix} = \begin{bmatrix} 2^{2-1}p(p-2) & 2^{2-2}p^2 \\ 2^{2-2}p^2 & 2^{2-1}p(p-2) \end{bmatrix}$$

$$* \, Z_{2^2}.Z_{2^2}^t = Z_{2^2}^t.Z_{2^2}$$

$$* \left| Z_{2^2} \right| = -2(p-2)^3 = -2^{2^2-3}(p-2)^{2^2-1}$$

$$* \ Z_{2^2}^{-1} = \frac{1}{-2(p-2)^3}\left[\begin{array}{cc|cc} -2(p-1)^3+3(p-1)^2-1 & (p-2)^2 & (p-2)^2 & (p-2)^2 \\ (p-2)^2 & -(p-2)^2 & (p-2)^2 & -(p-2)^2 \\ \hline (p-2)^2 & (p-2)^2 & -(p-2)^2 & -(p-2)^2 \\ (p-2)^2 & -(p-2)^2 & -(p-2)^2 & (p-2)^2 \end{array}\right]$$

$$* \ Z_{2^2}^{-1} = \frac{1}{-2^{2^2-3}(p-2)^{2^2-1}}\left[\begin{array}{cc|cc} -2(p-2)^3-3(p-2)^2 & (p-2)^2 & (p-2)^2 & (p-2)^2 \\ (p-2)^2 & -(p-2)^2 & (p-2)^2 & -(p-2)^2 \\ \hline (p-2)^2 & (p-2)^2 & -(p-2)^2 & -(p-2)^2 \\ (p-2)^2 & -(p-2)^2 & -(p-2)^2 & (p-2)^2 \end{array}\right]$$

$$* \ Z_{2^2}^{-1} = \frac{1}{-2^{2^2-3}(p-2)^{2^2-1}}\left[\begin{array}{c|c} (P-2)^2\overline{\overline{H_{2^1}}} & (P-2)^2 H_{2^1} \\ \hline (P-2)^2 H & (P-2)^2\overline{H_{2^1}} \end{array}\right]$$

Where $(p-2)^2\overline{\overline{H_{2^1}}}$ is the same $(P-2)^2 H_{2^1}$ after replacing the entry in the first row and first column by " $-2(p-2)^3-(2^2-1).(p-2)^2$.

In other words we find $B' = 2^{2^2-2-2}\,(p-2)^{2^2-2} H_{2^2}$, replacing the entry in first row and first column by " $-2^{2^2-3}(p-2)^{2^2-1}-(2^2-1).(p-2)^{2^2-2}$" we get the matrix B, after multiplying by $\frac{1}{-2^{2^2-3}(p-2)^{2^2-1}}$ we get $Z_{2^2}^{-1} = \frac{1}{-2^{2^2-3}(p-2)^{2^2-1}} B$.

iv. $$Z_{2^3} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & p-1 & 1 & p-1 & 1 & p-1 & 1 & p-1 \\ 1 & 1 & p-1 & p-1 & 1 & 1 & p-1 & p-1 \\ 1 & p-1 & p-1 & 1 & 1 & p-1 & p-1 & 1 \\ \hline 1 & 1 & 1 & 1 & p-1 & p-1 & p-1 & p-1 \\ 1 & p-1 & 1 & p-1 & p-1 & 1 & p-1 & 1 \\ 1 & 1 & p-1 & p-1 & p-1 & p-1 & 1 & 1 \\ 1 & p-1 & p-1 & 1 & p-1 & 1 & 1 & p-1 \end{array}\right]$$, then:

$* Z_{2^3} = \begin{bmatrix} Z_{2^2} & Z_{2^2} \\ Z_{2^2} & \overline{Z}_{2^2} \end{bmatrix}$, where " $\overline{Z}_{2^2} = (p-1)Z_{2^2}$ & $(p-1)^2 = 1$" or is the same Z_{2^2} after replacing, in it, each "1" by "(p-1)" and, in the same time, each "(p-1)" by "1".

$$* \ Z_{2^3}^t = Z_{2^3}, \ * Z_{2^3}.Z_{2^3}^t = \begin{bmatrix} 8 & 4p & 4p & 4p & \cdots & 4p \\ 4p & 8+4p(p-2) & 2p^2 & 2p^2 & \cdots & 2p^2 \\ 4p & 2p^2 & 8+4p(p-2) & 2p^2 & \cdots & 2p^2 \\ 4p & 2p^2 & 2p^2 & \ddots & \cdots & 2p^2 \\ \cdots & \cdots & \cdots & \cdots & 8+4p(p-2) & 2p^2 \\ 4p & 2p^2 & 2p^2 & \cdots & 2p^2 & 8+4p(p-2) \end{bmatrix}_{8\times 8}$$

$$Z_{2^3}.Z_{2^3}^t = 2^3 I_8 + \begin{bmatrix} 0 & 4p & 4p & 4p & \cdots & 4p \\ 4p & 4p(p-2) & 2p^2 & 2p^2 & \cdots & 2p^2 \\ 4p & 2p^2 & 4p(p-2) & 2p^2 & \cdots & 2p^2 \\ 4p & 2p^2 & 2p^2 & \ddots & \cdots & 2p^2 \\ \cdots & \cdots & \cdots & \cdots & 4p(p-2) & 2p^2 \\ 4p & 2p^2 & 2p^2 & \cdots & 2p^2 & 4p(p-2) \end{bmatrix}_{8\times 8}$$

$$Z_{2^3}.Z_{2^3}^t = 2^3 I_8 + \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}_{8\times 8}$$

$$A = \begin{bmatrix} 0 & 2^{3-1}p & 2^{3-1}p & 2^{3-1}p \\ 2^{3-1}p & 2^{3-1}p(p-2) & 2^{3-2}p^2 & 2^{3-2}p^2 \\ 2^{3-1}p & 2^{3-2}p^2 & 2^{3-1}p(p-2) & 2^{3-2}p^2 \\ 2^{3-1}p & 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-1}p(p-2) \end{bmatrix}_{4\times 4}$$

$$B = \begin{bmatrix} 2^{3-1}p & 2^{3-1}p & 2^{3-1}p & 2^{3-1}p \\ 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 \\ 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 \\ 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 \end{bmatrix}_{4\times 4}$$

$$C = \begin{bmatrix} 2^{3-1}p & 2^{3-1}p & 2^{3-1}p & 2^{3-1}p \\ 2^{3-2}p^2 & 2^{3-1}p(p-2) & 2^{3-2}p^2 & 2^{3-2}p^2 \\ 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-1}p(p-2) & 2^{3-2}p^2 \\ 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-2}p^2 & 2^{3-1}p(p-2) \end{bmatrix}_{4\times 4}$$

$$* \, Z_{2^3}.Z_{2^3}^t = Z_{2^3}^t.Z_{2^3}$$

$$* \left| Z_{2^3} \right| = -2^5(p-2)^7 = -2^{2^3-3}(p-2)^{2^3-1}$$

$$* \, Z_{2^3}^{-1} = \frac{1}{-2^5(p-2)^7} \left[\begin{array}{cc|cc|c} -2^5(p-2)^7 - (7)2^3(p-2)^6 & 2^3(p-2)^6 & 2^3(p-2)^6 & 2^3(p-2)^6 & \\ 2^3(p-2)^6 & -2^3(p-2)^6 & 2^3(p-2)^6 & -2^3(p-2)^6 & 2^3(P-2)^6 H_{2^2} \\ \hline 2^3(p-2)^6 & 2^3(p-2)^6 & -2^3(p-2)^6 & -2^3(p-2)^6 & \\ 2^3(p-2)^6 & -2^3(p-2)^6 & -2^3(p-2)^6 & 2^3(p-2)^6 & \\ \hline \multicolumn{4}{c|}{2^3(P-2)^6 H_{2^2}} & 2^3(P-2)^6 \overline{H_{2^2}} \end{array} \right]_{8\times 8}$$

$$* \, Z_{2^3}^{-1} = \frac{1}{-2^5(p-2)^7} \begin{bmatrix} 2^3(P-2)^6 \overline{\overline{H_{2^2}}} & 2^3(P-2)^6 H_{2^2} \\ 2^3(P-2)^6 H_{2^2} & 2^3(P-2)^6 \overline{H_{2^2}} \end{bmatrix}_{8\times 8}$$

Where the $2^3(p-2)^6\overline{\overline{H_{2^2}}}$ or is the same $2^3(p-2)^6 H_{2^2}$ after replacing the entry in the first row and first column by “ $-2^5(p-2)^7 - (7).2^3(p-2)^6$.

In other words we find $B' = 2^{2^3-3-2}(p-2)^{2^3-2} H_{2^3}$, replacing the entry in first row and first column by " $-\left(2^{2^3-3}\right)(p-2)^{2^3-1}-(2^3-1)2^{2^3-3-2}(p-2)^{2^3-2}$ ", we get the matrix *B* after that multiplying by $\dfrac{1}{-2^{2^3-3}(p-2)^{2^2-1}}$ we get $Z_{2^3}^{-1}=\dfrac{1}{-2^{2^3-3}(p-2)^{2^2-1}}B$.

v. * $Z_{2^4}=\begin{bmatrix} Z_{2^3} & Z_{2^3} \\ Z_{2^3} & \overline{Z}_{2^3} \end{bmatrix}$, where "$\overline{Z}_{2^3}=(p-1)Z_{2^3}$ & $(p-1)^2=1$" or is the same Z_{2^3} after replacing, in it, each "1" by "(*p*-1)" and, in the same time, each "(*p*-1)" by "1".

* $Z_{2^4}^t=Z_{2^4}$

* $Z_{2^4}.Z_{2^4}^t=Z_{2^4}^t.Z_{2^4}$

$$*\,Z_{2^4}.Z_{2^4}^t=\begin{bmatrix} 16 & 8p & 8p & 8p & \cdots & 8p \\ 8p & 16+8p(p-2) & 4p^2 & 4p^2 & \cdots & 4p^2 \\ 8p & 4p^2 & 16+8p(p-2) & 4p^2 & \cdots & 4p^2 \\ 8p & 4p^2 & 4p^2 & \ddots & \cdots & 4p^2 \\ \cdots & \cdots & \cdots & \cdots & 16+8p(p-2) & 4p^2 \\ 8p & 4p^2 & 4p^2 & \cdots & 4p^2 & 16+8p(p-2) \end{bmatrix}_{16\times16}$$

$$Z_{2^4}.Z_{2^4}^t=2^4I_{16}+\begin{bmatrix} 0 & 8p & 8p & 8p & \cdots & 8p \\ 8p & 8p(p-2) & 4p^2 & 4p^2 & \cdots & 4p^2 \\ 8p & 4p^2 & 8p(p-2) & 4p^2 & \cdots & 4p^2 \\ 8p & 4p^2 & 4p^2 & \ddots & \cdots & 4p^2 \\ \cdots & \cdots & \cdots & \cdots & 8p(p-2) & 4p^2 \\ 8p & 4p^2 & 4p^2 & \cdots & 4p^2 & 8p(p-2) \end{bmatrix}_{16\times16}$$

$$Z_{2^4}.Z_{2^4}^t=2^4I_{16}+\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}_{16\times16}$$

Where:

$$A=\begin{bmatrix} 0 & 2^3p & 2^3p & \cdots & \cdots & \cdots & 2^3p & 2^3p \\ 2^3p & 2^3p(p-2) & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 \\ 2^3p & 2^2p^2 & \ddots & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & 2^2p^2 & \ddots & 2^2p^2 & \cdots & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & \cdots & 2^2p^2 & \ddots & 2^2p^2 & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & \cdots & \cdots & 2^2p^2 & \ddots & 2^2p^2 & 2^2p^2 \\ 2^3p & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 & 2^3p(p-2) & 2^2p^2 \\ 2^3p & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^3p(p-2) \end{bmatrix}_{8\times8}$$

$$B=\begin{bmatrix} 2^3P & 2^3p & 2^3p & \cdots & \cdots & \cdots & 2^3p & 2^3p \\ 2^2p^2 & 2^2p^2 & 2^2p^2 & \cdots & \cdots & \cdots & \cdots & 2^2p^2 \\ 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & 2^2p^2 & \cdots & 2^2p^2 & \cdots & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & \cdots & 2^2p^2 & \cdots & 2^2p^2 & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & \cdots & \cdots & 2^2p^2 & \cdots & 2^2p^2 & 2^2p^2 \\ 2^2p^2 & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 & 2^2p^2 & 2^2p^2 \\ 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 \end{bmatrix}_{8\times8}$$

$$C=\begin{bmatrix} 2^3p(p-2) & 2^2p^2 & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 & 2^2p^2 \\ 2^2p^2 & \ddots & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 \\ 2^2p^2 & 2^2p^2 & \ddots & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & 2^2p^2 & \ddots & 2^2p^2 & \cdots & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & \cdots & 2^2p^2 & \ddots & 2^2p^2 & \cdots & 2^2p^2 \\ \vdots & 2^2p^2 & \cdots & \cdots & 2^2p^2 & \ddots & 2^2p^2 & 2^2p^2 \\ 2^2p^2 & 2^2p^2 & \cdots & \cdots & \cdots & 2^2p^2 & 2^3p(p-2) & 2^2p^2 \\ 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^2p^2 & 2^3p(p-2) \end{bmatrix}_{8\times 8}$$

$$*\ \left|Z_{2^4}\right| = -2^{2^4-3}(p-2)^{2^4-1}$$

$$*Z_{2^4}^{-1}=\frac{1}{-2^{2^4-3}(p-2)^{2^4-1}}\begin{array}{c} \begin{bmatrix} -2^{13}(p-2)^{15}-(2^4-1)2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} \\ 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & 2^{10}(p-2)^{14} & -2^{10}(p-2)^{14} \end{bmatrix}_{8\times 8} \\ Z'_{2^3} \qquad Z'_{2^3} \qquad -Z'_{2^3} \end{array}$$

$$Z'_{2^3}=2^{10}(p-2)^{14}H_{2^3},\ -Z'_{2^3}=\overline{Z}'_{2^3}=2^{10}(p-2)^{14}\overline{H_{2^3}}$$

Or

$$*\ Z_{2^4}^{-1}=\frac{1}{-2^{2^4-3}(P-2)^{2^4-1}}\begin{bmatrix} \overline{\overline{Z}}'_{2^3} & Z'_{2^3} \\ Z'_{2^3} & \overline{Z}'_{2^3} \end{bmatrix}$$

Where $\overline{\overline{Z}}'_{2^3}=Z'_{2^3}$ after replacing the entry in the first row and first column by

$$-2^{13}(p-2)^{15}-(2^4-1)2^{10}(p-2)^{14}.$$

In other words we find $B'=2^{2^4-4-2}(p-2)^{2^4-2}H_{2^4}$, replacing the entry in first row and first column by "$-2^{2^4-3}(p-2)^{2^4-2}-(2^4-1)2^{2^4-4-2}(p-2)^{2^4-2}$", we get the matrix B after multiplying by $\frac{1}{-2^{2^4-3}(p-2)^{2^4-1}}$ we get

$$Z_{2^4}^{-1}=\frac{1}{-2^{2^4-3}(p-2)^{2^4-1}}B.$$

vi. **Some properties of Z_m**

The useful some properties of standard Z_m, where $m=2^n, n\geq 2$ *or* $m=4n, n\geq 1$, except Z_1 and Z_2 (or corresponding with Hadamard matrix) are following:

a. $Z_{2^n}=\begin{bmatrix} Z_{2^{n-1}} & Z_{2^{n-1}} \\ Z_{2^{n-1}} & \overline{Z}_{2^{n-1}} \end{bmatrix}$, where "$\overline{Z}_{2^{n-1}}=(p-1)Z_{2^{n-1}}$ & $(p-1)^2=1$" or is the same $Z_{2^{n-1}}$ after replacing in it each "-1" by "(p-1)".
b. Any different two rows (columns) in Z_m are orthogonal by *mod p*.
c. Z_m is symmetric that is $Z_m=Z_m^t$
d. Each row (column) of Z_m contains *m* entries.
e. All entries in the first row (column) of Z_m are equals "1".

f. Each row (column) of Z_m except first row (column) contains $\frac{m}{2}$ of "1.s" and $\frac{m}{2}$ of "(p-1).s".

g. The distribution of "(p-1)" in Z_m is the same distribution of "-1" in H_m.

h. Any two different two rows (columns) except the first row(first column) contains $\frac{m}{2}$ of disagreements and $\frac{m}{2}$ of agreements, $\frac{m}{4}$ of agreements are "1.s" and $\frac{m}{4}$ of agreements are "(p-1).s".

i. For $i \neq j$ then the i^{th} row from Z_m and j^{th} column from H_m (except the first row in Z_m and first column in H_m) are orthogonal because for two different rows A_i, A_j from H_m and $i \neq j$ we have the distributions of "1" and "-1" :

$$\begin{array}{llllll} & \overbrace{\hspace{17em}}^{agre.\ m/2} & & \overbrace{\hspace{17em}}^{disag.\ m/2} \\ & \overbrace{1\ 1\ \dots\ 1}^{m/4} & \overbrace{-1-1\ \dots\ -1}^{m/4} & \overbrace{1\ \ 1\ \dots\ \ 1}^{m/4.} & \overbrace{-1-1\ \dots\ -1}^{m/4} \\ A_i & 1\ 1\ \dots\ 1 & -1-1\ \dots\ -1 & 1\ \ 1\ \dots\ \ 1 & -1-1\ \dots\ -1 \\ A_j & 1\ 1\ \dots 1 & -1-1\ \dots\ -1 & -1-1\ \dots\ -1 & 1\ \ 1\ \dots\ \ 1 \end{array}$$

Corresponding to "distribution of "1" and "-1" in the row A_i from H_m and distribution of "1" and (p-1) in the row B_j from Z_m " or "Corresponding to distribution of "1" and "-1" in the column A_i from H_m and distribution of "1" and (p-1) in the row B_j from Z_m"(H_m and Z_m are symmetric)

$$\begin{array}{lllll} & \overbrace{\hspace{20em}}^{agre.\ m/2} & & \overbrace{\hspace{20em}}^{disag. m/2} \\ & \overbrace{1\ 1\ \dots\ 1}^{m/4} & \overbrace{-1\quad -1\ \dots\quad -1}^{m/4} & \overbrace{1\quad 1\quad \dots\quad 1}^{m/4} & \overbrace{-1-1\ \dots\ -1}^{m/4} \\ A_i & 1\ 1\ \dots\ 1 & -1\quad -1\ \dots\quad -1 & 1\quad 1\quad \dots\quad 1 & -1-1\ \dots\ -1 \\ B_j & 1\ 1\ \dots 1 & p-1\ \ p-1\ \dots\ p-1 & p-1\ p-1\ \dots\ p-1 & 1\quad 1\ \dots\quad 1 \end{array}$$

Is very clear that A_i, B_j are orthogonal and B_i, A_j also orthogonal.

$$* Z_m . Z_m^t = Z_m^t . Z_m$$

Theorem 3: If Z_m is generalized Hadamard matrix in standard form of order *m* and *m* is power of 2 or not then:

$$Z_m Z_m^t = mI_m + \begin{bmatrix} 0 & (m/2)p & (m/2)p & \cdots & (m/2)p \\ (m/2)p & (m/2)p(p-1) & (m/4)p^2 & \cdots & (m/4)p^2 \\ (m/2)p & (m/4)p^2 & (m/2)p(p-1) & (m/4)p^2 \cdots & (m/4)p^2 \\ \vdots & \vdots & \begin{matrix}(m/4)p^2 \\ \vdots\end{matrix} & \ddots & \vdots \\ (m/2)p & (m/4)p^2 & (m/4)p^2 & (m/4)p^2 \cdots & (m/2)p(p-1) \end{bmatrix}_{m \times m} \qquad (6)$$

Suppose that $Z_m . Z_m^t = C = \left[c_{ij}\right]_{m \times m}$, C is symmetric, then:

- $c_{11} = m$
- $c_{1j} = \sum_{k=1}^{m} z_{1k} . z_{kj} = \sum_{\frac{m}{2}} 1 + \sum_{\frac{m}{2}} (p-1) = \sum_{\frac{m}{2}} p,\ j = 2,3,\dots,m$

$$c_{1j} = \frac{m}{2} p,\ j = 2,3,\dots,m \Rightarrow c_{i1} = \frac{m}{2} p, i = 2,3,\dots,m$$

- $c_{ii} = \sum_{k=1}^{m} z_{ik} . z_{ki} = \sum_{\frac{m}{2}} 1 + \sum_{\frac{m}{2}} (p-1)^2 , \quad i = 2,3,\ldots,m$

$$= \frac{m}{2} + \frac{m}{2}(p^2 - 2p + 1)$$

$$c_{ii} = m + \frac{m}{2} p(p-2), \quad i = 2,3,\ldots,m$$

- $c_{ij} = \sum_{\frac{m}{4}} 1 + \sum_{\frac{m}{4}} (p-1)^2 + \sum_{\frac{m}{2}} (p-1); i \neq j, i, j = 2,3,\ldots,m$

$$c_{ij} = \frac{m}{4} + \frac{m}{4}(p-1)^2 + \frac{m}{2}(p-1); i \neq j, i, j = 2,3,\ldots,m$$

$$c_{ij} = \frac{m}{4} p^2; i \neq j, i, j = 2,3,\ldots,m$$

Thus:

$$Z_m Z_m^t = mI_m + \begin{bmatrix} 0 & (m/2)p & (m/2)p & \cdots & (m/2)p \\ (m/2)p & (m/2)p(p-1) & (m/4)p^2 & \cdots & (m/4)p^2 \\ (m/2)p & (m/4)p^2 & (m/2)p(p-1) & (m/4)p^2 \cdots & (m/4)p^2 \\ \vdots & \vdots & (m/4)p^2 \; \vdots & \ddots & \vdots \\ (m/2)p & (m/4)p^2 & (m/4)p^2 & (m/4)p^2 \cdots & (m/2)p(p-1) \end{bmatrix}_{m \times m} \quad (7)$$

Thus, For $m = 2^n$:

$$* Z_{2^n} . Z_{2^n}^t = 2^n I_{2^n} + \begin{bmatrix} 0 & 2^{n-1}p & 2^{n-1}p & 2^{n-1}p & \cdots & 2^{n-1}p \\ 2^{n-1}p & 2^{n-1}p(p-2) & 2^{n-2}p^2 & 2^{n-2}p^2 & \cdots & 2^{n-2}p^2 \\ 2^{n-1}p & 2^{n-2}p^2 & 2^{n-1}p(p-2) & 2^{n-2}p^2 & \cdots & 2^{n-2}p^2 \\ 2^{n-1}p & 2^{n-2}p^2 & 2^{n-2}p^2 & \ddots & \cdots & 2^{n-2}p^2 \\ \cdots & \cdots & \cdots & \cdots & 2^{n-1}p(p-2) & \cdots \\ 2^{n-1}p & 2^{n-2}p^2 & 2^{n-2}p^2 & \cdots & \cdots & 2^{n-1}p(p-2) \end{bmatrix}_{2^n \times 2^n}$$

***Theorem* 4**: If Z_n in the standard form then:

1. $\left| Z_{2^n} \right| = -\left(2^{2^n - 3}\right)(p-2)^{2^n - 1}$

2. $Z_{2^n}^{-1} = \frac{1}{-\left(2^{2^n-3}\right)(p-2)^{2^n-1}} \begin{bmatrix} Z''_{2^{n-1}} & Z'_{2^{n-1}} \\ Z'_{2^{n-1}} & -Z'_{2^{n-1}} \end{bmatrix}_{2^n \times 2^n}$ *and*

$$* \; Z'_{2^{n-1}} = \begin{bmatrix} 2^{2^n-n-2}(p-2)^{2^n-2} & 2^{2^n-n-2}(p-2)^{2^n-2} & 2^{2^n-n-2}(p-2)^{2^n-2} & \cdots & 2^{2^n-n-2}(p-2)^{2^n-2} \\ 2^{2^n-n-2}(p-2)^{2^n-2} & \pm 2^{2^n-n-2}(p-2)^{2^n-2} & \pm 2^{2^n-n-2}(p-2)^{2^n-2} & \cdots & \pm 2^{2^n-n-2}(p-2)^{2^n-2} \\ 2^{2^n-n-2}(p-2)^{2^n-2} & \pm 2^{2^n-n-2}(p-2)^{2^n-2} & \pm 2^{2^n-n-2}(p-2)^{2^n-2} & \cdots & \pm 2^{2^n-n-2}(p-2)^{2^n-2} \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 2^{2^n-n-2}(p-2)^{2^n-2} & \pm 2^{2^n-n-2}(p-2)^{2^n-2} & \pm 2^{2^n-n-2}(p-2)^{2^n-2} & \cdots & \pm 2^{2^n-n-2}(p-2)^{2^n-2} \end{bmatrix}_{2^{n-1} \times 2^{n-1}}$$

Where the signals + or – of the entries in $Z'_{2^{n-1}}$ corresponding to the signals of entries in Hadamard matrix of the same size, and $Z''_{2^{n-1}}$ is the same $Z'_{2^{n-1}}$ after replacing the entry in the first row and first column in $Z'_{2^{n-1}}$ by: $-2^{2^n-3}(p-2)^{2^n-1}-(2^n-1)2^{2^n-n-2}(p-2)^{2^n-2}$.

In other words we find the matrix: $B'_{2^n}=2^{2^n-n-2}(p-2)^{2^n-2}H_{2^n}$ and after replacing the entry in first row and first column in it by: $-2^{2^n-3}(p-2)^{2^n-1}-(2^n-1)2^{2^n-n-2}(p-2)^{2^n-2}$ we get the matrix B then multiplying B by $\frac{1}{-\left(2^{2^n-3}\right)(p-2)^{2^n-1}}$ and we get

$$Z_{2^n}^{-1}=\frac{1}{-\left(2^{2^n-3}\right)(p-2)^{2^n-1}}B\ .$$

First: For $m=2^n$, $Z_{2^n}=A=\begin{bmatrix}A_1\\A_2\\\vdots\\A_m\end{bmatrix}$ and the matrix $2^{2^n-n-2}(p-2)^{2^n-2}H_{2^n}=B'=\left[B'_1,B'_2,\cdots,B'_m\right]$

after replacing the entry in the first row and first column in $2^{2^n-n-2}(p-2)^{2^n-2}H_{2^n}$ by

$\left[-\left(2^{2^n-3}\right)(p-2)^{2^n-1}-\left(2^n-1\right)2^{2^n-n-2}(p-2)^{2^n-2}\right]$ we get $B=\left[B_1,B_2,\cdots,B_m\right]$ and $A.B=C=\left[c_{ij}\right]_{m\times m}$ then we have:

$$*c_{11}=A_1.B_1=\left[-\left(2^{2^n-3}\right)(p-2)^{2^n-1}-\left(2^n-1\right)2^{2^n-n-2}(p-2)^{2^n-2}\right]+\left(2^n-1\right)2^{2^n-n-2}(p-2)^{2^n-2}$$

$$c_{11}=-\left(2^{2^n-3}\right)(p-2)^{2^n-1}$$

From the property i.

- $c_{1j}=2^{2^n-n-2}(p-2)^{n-2}\left(\sum_{i=1}^{m}b_{ij}\right)=0,\ j=1,3,\dots,m$

  $c_{i1}=0,\ i=2,3,\dots,n$
- $c_{ii}=A_2.B_2=2^{2^n-n-2}(p-2)^{n-2}\left[\sum_{2^{n-1}}1-\sum_{2^{n-1}}(p-1)\right],\ i=2,3,\dots,m$

  $c_{ii}=2^{2^n-n-2}(p-2)^{n-2}\left[\sum_{2^{n-1}}1-\sum_{2^{n-1}}(p-2)-\sum_{2^{n-1}}1\right],\ i=2,3,\dots,m$

  $c_{ii}=2^{2^n-n-2}(p-2)^{n-2}\left[-2^{n-1}(p-2)\right],\ i=2,3,\dots,m$

  $c_{ii}=-2^{2^n-3}(p-2)^{n-1},\ i=2,3,\dots,m$
- $c_{ij}=A_i.B_j=0,\ i\neq j;\ i,j=2,3,\dots,m-1$ this is from property *h*.

Thus:

$$\frac{1}{-\left(2^{2^n-3}\right)(p-2)^{2^n-1}}Z_{2^n}B=I_{2^n}$$

Or:

$$Z_{2^n}^{-1} = \frac{1}{-\left(2^{2^n-3}\right)(p-2)^{2^n-1}} B \qquad (8)$$

First finding $B' = 2^{2^n-n-2}(p-2)^{2^n-2} H_{2^n}$, after replacing the entry in first row and first column in $2^{2^n-n-2}(p-2)^{2^n-2} H_{2^n}$ by $\left[-\left(2^{2^n-3}\right)(p-2)^{2^n-1} - \left(2^n-1\right)2^{2^n-n-2}(p-2)^{2^n-2}\right]$ then we get *B* and after we find $Z_{2^n}^{-1}$. H_m, Z_m Note that H_m, Z_m are written in the standard form.

And;

* $\dfrac{1}{-\left(2^{2^n-3}\right)(p-2)^{2^n-1}} = \dfrac{1}{-\left(2^{m-3}\right)(p-2)^{m-1}}$

* $2^{2^n-n-2}(p-2)^{2^n-2} H_{2^n} = \dfrac{2^{2^n-2}}{2^n}(p-2)^{2^n-2} H_{2^n}$

$$2^{2^n-n-2}(p-2)^{2^n-2} H_{2^n} = \frac{2^{m-2}}{m}(p-2)^{m-2} H_m$$

- $\left[-\left(2^{2^n-3}\right)(p-2)^{2^n-1} - \left(2^n-1\right)2^{2^n-n-2}(p-2)^{2^n-2}\right] = \left[-\left(2^{m-3}\right)(p-2)^{m-1} - (m-1)\dfrac{2^{m-2}}{m}(p-2)^{m-2}\right]$

And :

$$Z_m^{-1} = \frac{1}{-\left(2^{m-3}\right)(p-2)^{m-1}} B$$

First we find $B' = \dfrac{2^{m-2}}{m}(p-2)^{m-2} H_m$ after replacing the entry in the first row and first column in $B' = \dfrac{2^{m-2}}{m}(p-2)^{m-2} H_m$ by $\left[-\left(2^{m-3}\right)(p-2)^{m-1} - (m-1)\dfrac{2^{m-2}}{m}(p-2)^{m-2}\right]$ and we get *B* and after we find Z_m^{-1}.

***Example* 1:** For the following Hadamard H'_{12}

$$H'_{12} = \begin{bmatrix}
1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - \\
1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - \\
- & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - \\
- & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 \\
1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 \\
- & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 \\
- & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - \\
1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 \\
- & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1
\end{bmatrix}$$

Rewriting H'_{12} in the standard Hadamard form H_{12} :

$$H_{12}=\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&-&-&1&-&-&1&-&-&1\\
1&1&1&-&1&-&-&1&-&-&1&-\\
1&-&-&-&1&1&1&1&1&-&-&-\\
1&-&1&1&-&-&1&1&-&-&-&1\\
1&1&-&1&-&-&1&-&1&-&1&-\\
1&-&-&-&-&-&-&1&1&1&1&1\\
1&-&1&-&-&1&1&-&-&1&1&-\\
1&1&-&-&1&-&1&-&-&1&-&1\\
1&-&-&1&1&1&-&-&-&-&1&1\\
1&-&1&1&1&-&-&-&1&1&-&-\\
1&1&-&1&-&1&-&1&-&1&-&-
\end{bmatrix}$$

Corresponding to standard Z_{12} :

$$Z_{12}=\begin{bmatrix}
1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&(p-1)&(p-1)&1&(p-1)&(p-1)&1&(p-1)&(p-1)&1\\
1&1&1&(p-1)&1&(p-1)&(p-1)&1&(p-1)&(p-1)&1&(p-1)\\
1&(p-1)&(p-1)&(p-1)&1&1&1&1&1&(p-1)&(p-1)&(p-1)\\
1&(p-1)&1&1&(p-1)&(p-1)&1&1&(p-1)&(p-1)&(p-1)&1\\
1&1&(p-1)&1&(p-1)&(p-1)&1&(p-1)&1&(p-1)&1&(p-1)\\
1&(p-1)&(p-1)&(p-1)&(p-1)&(p-1)&(p-1)&1&1&1&1&1\\
1&(p-1)&1&(p-1)&(p-1)&1&1&(p-1)&(p-1)&1&1&(p-1)\\
1&1&(p-1)&(p-1)&1&(p-1)&1&(p-1)&(p-1)&1&(p-1)&1\\
1&(p-1)&(p-1)&1&1&1&(p-1)&(p-1)&(p-1)&(p-1)&1&1\\
1&(p-1)&1&1&1&(p-1)&(p-1)&(p-1)&1&1&(p-1)&(p-1)\\
1&1&(p-1)&1&(p-1)&1&(p-1)&1&(p-1)&1&(p-1)&(p-1)
\end{bmatrix}_{12\times 12}$$

For H_{12} we have (m/2) = 6 and (m/4) = 3 and:

$$Z_{12}.Z_{12}^{t}=12I_{12}+\begin{bmatrix}
0&6p&6p&6p&\cdots&\cdots&6p\\
6p&6p(p-1)&3p^2&3p^2&\cdots&\cdots&3p^2\\
6p&3p^2&6p(p-1)&3p^2&\cdots&\cdots&3p^2\\
6p&3p^2&3p^2&6p(p-1)&3p^2&\cdots&3p^2\\
\vdots&3p^2&\vdots&3p^2&\ddots&\cdots&3p^2\\
\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\
6p&3p^2&3p^2&\cdots&\cdots&3p^2&6p(p-1)
\end{bmatrix}_{12\times 12}$$

And;

$$Z_m^{-1}=\frac{1}{-\left(2^{m-3}\right)\left(p-2\right)^{m-1}}B$$

First finding:

$$B' = \frac{2^{m-2}}{m}(p-2)^{m-2}H_{12} = \frac{2^{10}}{12}(P-2)^{10}\begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&-&-&1&-&-&1&-&-&1\\ 1&1&1&-&1&-&-&1&-&-&1&-\\ 1&-&-&-&1&1&1&1&1&-&-&-\\ 1&-&1&1&-&-&1&1&-&-&-&1\\ 1&1&-&1&-&-&1&-&1&-&1&-\\ 1&-&-&-&-&-&-&1&1&1&1&1\\ 1&-&1&-&-&1&1&-&-&1&1&-\\ 1&1&-&-&1&-&1&-&-&1&-&1\\ 1&-&-&1&1&1&-&-&-&-&1&1\\ 1&-&1&1&1&-&-&-&1&1&-&-\\ 1&1&-&1&-&1&-&1&-&1&-&- \end{bmatrix}$$

After replacing the entry in the first row and first column in B' by $\left[-(2)^9(p-2)^{11}-(11)\frac{2^{10}}{12}(p-2)^{10}\right]$

Thus:

$$Z_{12}^{-1} = \frac{1}{-\left(2^9\right)\left(p-2\right)^{11}}\begin{bmatrix} -(2)^9(p-2)^{11}-(11)\frac{2^{10}}{12}(p-2)^{10} & \pm\frac{2^{10}}{12}(p-2)^{10} & \cdots & \pm\frac{2^{10}}{12}(p-2)^{10}\\ \pm\frac{2^{10}}{12}(p-2)^{10} & \pm\frac{2^{10}}{12}(p-2)^{10} & \cdots & \pm\frac{2^{10}}{12}(p-2)^{10}\\ \vdots & \vdots & \vdots & \vdots\\ \pm\frac{2^{10}}{12}(p-2)^{10} & \pm\frac{2^{10}}{12}(p-2)^{10} & \cdots & \pm\frac{2^{10}}{12}(p-2)^{10} \end{bmatrix}_{12\times 12}$$

Where the distribution of + and – in the matrix is the same distribution in H_{12} and We can check that $Z_{12}.Z_{12}^{-1} = I_{12}$.

vii. Compose two generalized Hadamard matrices: let's Z_n and Z_m two generalized Hadamard matrices of orders n, m respectively and $\overline{Z_m}$ is Z_m after replacing each "1" in Z_m by (p-1) and each (p-1) in Z_m by "1' or $\overline{Z_m}$ =(p-1) Z_m and after replacing each $(p-1)^2$ by"1' then compose Z_n with Z_m or $Z_n(Z_m)$ is a generalized Hadamard matrices of order

$n \times m$.

Example 2**:** Find $Z_{2^2}(Z_2)$

$$Z_{2^2} = \begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & (p-1) & 1 & (p-1)\\ 1 & 1 & (p-1) & (p-1)\\ 1 & (p-1) & (p-1) & 1 \end{bmatrix},\ Z_2 = \begin{bmatrix} 1 & 1\\ 1 & (P-1) \end{bmatrix}, \overline{Z_2} = \begin{bmatrix} (p-1) & (p-1)\\ (p-1) & 1 \end{bmatrix}$$

$$Z_{2^2}(Z_2)=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & (P-1) & 1 & (P-1) & 1 & (P-1) & 1 & (P-1) \\ 1 & 1 & (p-1) & (p-1) & 1 & 1 & (p-1) & (p-1) \\ 1 & (P-1) & (p-1) & 1 & 1 & (P-1) & (p-1) & 1 \\ 1 & 1 & 1 & 1 & (p-1) & (p-1) & (p-1) & (p-1) \\ 1 & (P-1) & 1 & (P-1) & (p-1) & 1 & (p-1) & 1 \\ 1 & 1 & (p-1) & (p-1) & (p-1) & (p-1) & 1 & 1 \\ 1 & (P-1) & (p-1) & 1 & (p-1) & 1 & 1 & (P-1) \end{bmatrix}=Z_{2^3}$$

By the same way $Z_4(Z_{12})=Z_{48}$

***viii.* Binary Representation of Standard** Z_m: Suppose Z_m is a standard generalized Hadamard matrix, replacing each "1" by "0" and in the same time each "(p-1)' by "1" then we have the binary representation of Z_m.

The rows (columns) in binary representation of standard generalized Hadamard matrix Z_m form an additive group where the addition is done by *mod* 2.

Example 3: the binary representation of Z_{2^2} is:

$$Z_{2^2}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (p-1) & 1 & (p-1) \\ 1 & 1 & (p-1) & (p-1) \\ 1 & (p-1) & (p-1) & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Conclusion

1. Generalized Hadamard matrix Z_m of order *m* based on the Hadamard matrix H_m of the same order after replacing each "-1" in H_m by "(*p*-1)" and *m* is 1,2 or 4*n* where *n* is positive integer.
2. Z_m is symmetric and each two different rows (columns) are orthogonal by *mod p*.
3. All the entries in the first row(column) in the standard Generalized Hadamard matrix Z_m are "1.s" and each other row (column) contains *m*/2 of "1.s" and *m*/2 of "(*p*-1).
4. Any two different rows (columns) of Z_m except the first row(column) contain *m*/2 of disagreements and *m*/2 of agreements and, *m*/4 of agreements are "1.s" and the same number of agreements are "(*p*-1).s" and from this *m* is 1,2 or 4*n* .
5. Any two different rows Z_i and Z_j (and any two different row Z_i and column Z_j that is $i \neq j$) from Z_m except the first row and first column are orthogonal by *mod p*.
6. If Z_m is generalized Hadamard matrix then $Z_{2m}=\begin{bmatrix} Z_m & Z_m \\ Z_m & \overline{Z_m} \end{bmatrix}$ is also generalized Hadamard matrix where $\overline{Z_m}$ is $(p-1)Z_m$ after replacing each (*p*-1)^2 by "1".
7. If Z_m is generalized Hadamard matrix in standard form of order *m* and *m* is power of 2 0r
8. not

then: $$Z_m Z_m^t = mI_m + \begin{bmatrix} 0 & (m/2)p & (m/2)p & \cdots & (m/2)p \\ (m/2)p & (m/2)p(p-1) & (m/4)p^2 & \cdots & (m/4)p^2 \\ (m/2)p & (m/4)p^2 & (m/2)p(p-1) & (m/4)p^2 \cdots & (m/4)p^2 \\ \vdots & \vdots & (m/4)p^2 \ \vdots & \ddots & \vdots \\ (m/2)p & (m/4)p^2 & (m/4)p^2 & (m/4)p^2 \cdots & (m/2)p(p-1) \end{bmatrix}$$

9. If Z_m is generalized Hadamard matrix in standard form of order *m* and *m* is power of 2 0r not

then: $Z_m^{-1} = \frac{1}{-\left(2^{m-3}\right)\left(p-2\right)^{m-1}} B$

First we find $B' = \frac{2^{m-2}}{m}(p-2)^{m-2} H_m$ after replacing the entry in the first row and first in

$$B' = \frac{2^{m-2}}{m}(p-2)^{m-2} H_m$$ by

$\left[-\left(2^{m-3}\right)(p-2)^{m-1} - (m-1)\frac{2^{m-2}}{m}(p-2)^{m-2}\right]$ and we get *B* and after we find Z_m^{-1}.

10. If Z_n, Z_m are two standard generalized Hadamard matrices then the compose $Z_n(Z_m)$ and $Z_m(Z_m)$ are generalized Hadamard matrices of order $n \times m$.

11. The binary representation of standard Hadmard matrix Z_m forms an additive group where the addition is done by mode 2.

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