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Compose Binary Matrices

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ABSTRACT

Hadamard Matrices and M-Sequences (which formed a closed sets under the addition and with the corresponding null sequence formed additive groups and generated by feedback registers) are used widely at the forward links of communication channels to mix the information on connecting to and at the backward links of these channels to sift through this information is transmitted to reach the receivers this information in correct form, specially in the pilot channels, the Sync channels, and the Traffic channel.

This research is useful to generate new sets of sequences (which are also with the corresponding null sequence additive groups) by compose Hadamard matrices and M-sequences with the bigger lengths and the bigger minimum distance that assists to increase secrecy of these information and increase the possibility of correcting mistakes resulting in the channels of communication.

Keywords: hadamard matrices, Walsh Sequences, M-sequences, Additive group, Coefficient of Correlation, Orthogonal sequences.

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Introduction

- ***M- Sequences***: Let *k* be a positive integer and $\lambda, \lambda_0, \lambda_1, ..., \lambda_{k-1}$ are elements in the field F_2, then the sequence $a_0, a_1, ...$ is called non homogeneous linear recurring sequence of order *k iff* **:**

$$a_{n+k} = \lambda_{k-1}a_{n+k-1} + \lambda_{k-2}a_{n+k-2} + ... + \lambda_0 a_n + \lambda,$$
$$\lambda_i \in F_2,\ i = 0,1,...,k-1$$
$$or \qquad a_{n+k} = \sum_{i=0}^{k-1} \lambda_i a_{n+i} + \lambda \qquad (1)$$

The elements $a_0, a_1, ..., a_{k-1}$ are called the initial values (or the vector $(a_0, a_1, ..., a_{k-1})$ is called the initial vector). If $\lambda = 0$ then the sequence $a_0, a_1, ...$ is called homogeneous linear recurring sequence (H. L. R. S.), except the zero-initial vector, and the polynomial

$$f(x) = x^k + \lambda_{k-1}x^{k-1} + ... + \lambda_1 x + \lambda_0 \qquad (2)$$

is called the characteristic polynomial. In this study, we are limited to $\lambda_0 = 1$.

If the characteristic polynomial is prime then the sequence $a_0, a_1, ...$ is called M-Sequence and

this sequence is periodic with period $n = 2^k - 1$, and each period contains $n_1 = 2^{k-1} - 1$ of "0.s"

and $n_2 = 2^{k-1}$ of "1.s".

The set of all $n = 2^k - 1$ cyclic permutations of one period is closed under the addition by *mod* 2 and form an orthogonal set and any two different permutations contain 2^{k-1} of disagreements and $(2^{k-1} - 1)$ of agreements, $f_1 = (2^{k-2} - 1)$ of the agreements are "0.s" and $f_2 = 2^{k-2}$ of the agreements are "1.s".

We can collect the all permutations of one period in one square matrix as following:

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & \cdots & a_{n-1} & a_0 \end{bmatrix}$$

Example 1: If α is a root of the prime polynomial $f(x) = x^2 + x + 1$ and generates $GF(2^2)$ and Suppose the Linear Binary Recurring Sequence be

$a_{n+2} = a_{n+1} + a_n$ or $a_{n+2} + a_{n+1} + a_n = 0$

With the characteristic equation $x^2 + x + 1 = 0$ and the characteristic polynomial $f(x) = x^2 + x + 1$, which is a prime then the general solution of equation For the initial position: $a_1 = 1, a_2 = 0$ is given by: $a_n = \alpha \cdot \alpha^n + \alpha^2 \cdot \alpha^{2n}$, and the sequence is periodic with the period $2^2 - 1 = 3$ and x_1= (101), by the cyclic permutations on x_1 we have $M_3 = \{x_1, x_2, x_3\}$ where:

x_1= (101), x_2 =(110), x_3=(011), The first two digits in each sequence are the initial position of the feedback register, and the set M_3 is an orthogonal set. The matrix of the all cyclic permutations is:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$n = 2^2 - 1 = 3. \quad n_1 = 2^{2-1} - 1 = 1, \quad n_2 = 2^{2-1} = 2,$$
$$f_1 = 2^{2-1} - 1 = 1, \quad f_2 = 2^{2-1} = 2$$

Example 2: **T**he prime polynomial $f(x) = x^3 + x + 1$ is prime and generates $GF(2^3)$ Suppose the Linear Recurring Sequence be :

$a_{n+3} = a_{n+1} + a_n$ or $a_{n+3} + a_{n+1} + a_n = 0$

The characteristic equation $x^3 + x + 1 = 0$ and the characteristic polynomial $f(x) = x^3 + x + 1$, which is a prime and generates F_{2^3} and for the initial position: $a_1 = 1, a_2 = 0$, $a_3 = 0$, and the sequence is periodic with the period $2^3 - 1 = 7$ and y_1 = (1001011) , by the cyclic permutations on y_1 we have

$M_7 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$ where: y_2 = (1100101) , y_3 = (1110010),

y_4 = (0111001), y_5 =(1011100), y_6 =(0101110), y_7 =(0010111), the first three digits in each

sequence are the initial position of the feedback register, and the set M_7 is an orthogonal set. The matrix of the all cyclic permutations is:

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$m = 2^3 - 1 = 7. \quad m_1 = 2^{3-1} - 1 = 3, \quad n_2 = 2^{3-1} = 4,$$

$$g_1 = 2^{3-1} - 1 = 3, \quad f_2 = 2^{3-1} = 4$$

[7], [10], [12], [15-18], [20], [27]

- ***Hadmard Matrices***: Hadamard matrices seem such simple matrix structures: they are square, have entries +1 or −1 and have orthogonal row vectors and orthogonal column vectors.

A Hadamard matrix is invented by Sylvester (1867), 26 years before Hadamard (1893) considered them. The $n \times n$ Hadamard matrix H_n must have $n(n-1)/2$ of "-1.s" and $n(n+1)/2$ of "1.s". The binary representation of Hadamard matrix gets replaced each"1" by "0" and replaced each "-1" by"1"

Some basic properties of Hadamard matrices are given by following theorem:

Let H_h be an Hadamard matrix of order *h* then:

1. $H_h H_h^t = hI_h$, where I_h is the identity matrix of order *h*;
2. $\left|\det H\right| = h^{\frac{1}{2}h}$;
3. $H_h H_h^t = H_h^t H_h$;
4. Hadamard matrices may be changed into other Hadamard matrices by permuting rows and columns and by multiplying rows and columns by −1.
5. Matrices which can be obtained from one another by these methods are referred to as *H*-equivalent (not all Hadamard matrices of the same order are *H*-equivalent).
6. Every Hadamard matrix is *H*-equivalent to an Hadamard matrix which has every element of its first row and column equal +1 – matrices of this latter form are called normalized.
7. If H_{4n} is a normalized Hadamard matrix of order 4*n*, then every row (column), except the first, has 2*n* minus ones and 2*n* plus ones in each row (column), and the set these rows is called Walsh's sequences
8. The order of an Hadamad matrix is 1, 2, 4*n*, where *n* is a positive integer.

(Sylvester) Let H_1 and H_2 be two Hadamard matrices of orders h_1 and h_2 , then the Kronecker product of H_1 and H_2 is an Hadamard matrix of order $h_1 h_2$.

9. Standard Hadamard matrix H_h have orthogonal rows (columns) vectors and each row (column) except the first row and first column contains h/2 of "1.s" and h/2 of "-1" and h/2 of disagreements and h/2 of agreements h/4 of agreements are "1.s" and h/4 of agreements are "-1". [1-6], [8,9], [13,14], [19-26], [28,32].

The smallest examples are:

$$H_{2^0} = [1], \quad H_{2^1} = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}, \quad H_{2^2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix},$$

$$H_{2^3} = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ \hline 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{array}\right], \quad H_{12} = \begin{bmatrix} 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 \\ 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - \\ 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & - & - \\ - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - \\ - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 \\ 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 \\ - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 \\ - & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - \\ 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 \\ - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 \end{bmatrix}$$

Research method and Material

***Definition* 1.** The Ultimately Periodic Sequence $a_0, a_1, ...$ with the smallest period *r* is called a periodic *iff*: $a_{n+r} = a_n$; $n = 0, 1, ...$ [1,2], [12], [15-18], [20-27]

***Definition* 2.** The complement of the binary vector $X = (x_1, x_2, ..., x_n),\ x_i \in F_2\{0,1\}$ is the vector $\overline{X} = (\overline{x_1}, \overline{x_2}, ..., \overline{x_n})$, where:

$$\overline{x_i} = \begin{cases} 1 & if \quad x_i = 0 \\ 0 & if \quad x_i = 1 \end{cases}. \ [1,2] \qquad (3)$$

***Definition* 3.** Suppose $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0, y_1, ..., y_{n-1})$ are binary vectors of length n on *F*2={0,1}. The coefficient of correlations function of *x* and *y* , denoted by $R_{x,y}$, is:

$$R_{x,y} = \sum_{i=0}^{n-1} (-1)^{x_i + y_i} \qquad (4)$$

Where $x_i + y_i$ is computed *mod* 2. It is equal to the number of agreements components minus the number of disagreements corresponding to components or if $x_i, y_i \in \{1, -1\}$ (usually, replacing in binary vectors *x* and *y* each "1" by "-1" and each "0" by "1") then

$$R_{x,y} = \sum_{i=0}^{n-1} x_i y_i \text{ , [1,2],[13-18]} \qquad (5)$$

***Definition* 4.** Any Periodic Sequence $a_0, a_1, ...$ over F_2 with prime characteristic polynomial is an orthogonal cyclic code and ideal auto correlation [1-2], [12], [18].

***Definition* 5.** Suppose $x = (x_0, x_1, ..., x_{n-1})$ and $y = (y_0, y_1, ..., y_{n-1})$ are binary vectors of length *n* on *GF*(2)={0,1}, or components belong to {1, -1}, is said strictly orthogonal if $R_{x,y} = 0$ and orthogonal if $R_{x,y} \in \{-1, 0.1\}$). [1,2],[13-18]

***Definition* 6.** Suppose *G* is a set of binary vectors of length *n*:

$$G = \begin{cases} X ; X = (x_0, x_1, ..., x_{n-1}), \\ x_i \in F_2 = \{0,1\}, i = 0,1,..., n-1 \end{cases}$$

Let's $1^* = -1$ and $0^* = 1$, The set *G* is said to be strictly orthogonal if the following two conditions are satisfied:

$$1.\ \forall X \in G, \sum_{i=0}^{n-1} x_i^* = 0, or \left|R_{x,0}\right| = 0. \qquad (6)$$

$$2.\ \forall X, Y \in G (X \neq Y), \sum_{i=0}^{n-1} x_i^* y_i^* = 0\, or \left|R_{x,y}\right| = 0. \quad (7)$$

That is, the absolute value of "the number of agreements minus the number of disagreements" is equal to zero, and orthogonal if the following two conditions are satisfied:

$$1.\ \forall X \in G, \sum_{i=0}^{n-1} x_i^* \leq 1, or \left|R_{x,0}\right| \leq 1. \qquad (8)$$

$$2.\ \forall X, Y \in G (X \neq Y), \sum_{i=0}^{n-1} x_i^* y_i^* \leq 1 or \left|R_{x,y}\right| \leq 1. \quad (9)$$

That is, the absolute value of "the number of agreements minus the number of disagreements" is equal to zero. [1,2],[13-18]

***Definition* 7.** The matrix $A = [a_{ij}]$ is called Hadamard matrix if it is a square matrix and each entry is equals 1 or "-" (where - denotes -1) with the property that if the size of *A* is then $A^T A = A A^T = h I_h$, in the decimal counting system, and the distinct rows vectors are mutually orthogonal. [20-27]

***Definition* 8.** If all entries of the first row and the first column in the Hadamard matrix are equal to "1" then the matrix is called standard Hadamard matrix.

Such matrices were first invented by Sylvester (1867) who observed that if *H* is an Hadamard matrix, then:

$$H = \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \qquad (10)$$

Is also an Hadamard matrix. [3-12]

***Theorem* 1.**

i. If $a_0, a_1, ...$ is a homogeneous linear recurring sequence of order *k* in F_2 , satisfies

ii. (1) then this sequence is periodic

iii. If this sequence is homogeneous linear recurring sequence, periodic with the period *r*, and its characteristic polynomial $f(x)$ then $r \mid ord\ f(x)$.

iv. If the polynomial $f(x)$ is primitive then the period of the sequence is $2^k - 1$, and this sequence is called M – sequence. [7], [16], [17], [27]

***Lema* 2**. (Sylvester (1867)): There is an Hadamard matrix of order 2^t for all nonnegative *t*. The matrices of order 2^t constructed using Sylvester's construction are usually Referred to as Sylvester-Hadamard matrices. [3-6], [13,15], Hadamard (1983) gave examples for a few small orders. [5-6],[8],[13-14],[22-23],[31-32]

Results and Discussion

First. Suppose *A* and *B* are tow square binary matrices of orders *n*, *m* respectively, compose the matrix *A* with the matrix *B* or *A*(*B*) is a the result of replacing each entry "0" in the matrix *A* by the full matrix *B* and replacing each "1" in the matrix *A* by full $\overline{B}$ complement of the matrix *B* and *A*(*B*) is a square matrix of order *n.m* .

The study is similar when *A* or *B* is rectangular matrix.

***Example* 3.** If :

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

then: $\overline{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

and: $A(B) = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$

If: $A = [a_{ij}]_n = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$,

$$B = [b_{kl}]_m = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix} \Rightarrow$$

$$A(B) = \begin{bmatrix} A_1(B_1) \\ A_1(B_2) \\ \cdots \\ A_1(B_m) \\ \hline A_2(B_1) \\ A_2(B_2) \\ \cdots \\ A_2(B_m) \\ \hline \vdots \\ A_n(B_1) \\ A_n(B_2) \\ \cdots \\ A_n(B_m) \end{bmatrix}_{n.m} \qquad (11)$$

And $A_{ij} = A_i(B_j)$ the result of compose the *i*^th^ row in the matrix *A* with the *j*^th^ row of the matrix *B*.

If A_i contains n_1 of "0.s" and n_2 of "1.s" then $\overline{A_i}$ contains n_1 of "1.s" and n_2 of "0.s" and if B_j contains m_1 of "0.s" and m_2 of "1.s" then $\overline{B_j}$ contains m_1 of "1.s" and m_2 of "0.s" and $A_{ij} = A_i(B_j)$ contains $n_1 m_1 + n_2 m_2$ of '0.s' and $n_1 m_2 + n_2 m_1$ of "1.s" and $\overline{A}_{ij} = \overline{A}_i(\overline{B}_j)$ contains

The same number of '0.s' and of "1.s", from example 1.:

$A_{23} = A_2(B_3) = [1\,0]([1\,0\,1]) = [0\,1\,0\,1\,0\,1]$.

Our purpose now to study the following: From the information about two rows A_{i_1}, A_{i_2} from the matrix *A* and two rows B_{j_1}, B_{j_2} from the matrix *B* then what is the number of agreements and the number of disagreements between $A_{i_1}(B_{j_1})$ *and* $A_{i_2}(B_{j_2})$.

If there are two rows from the matrix *A*, for simplicity A_1, A_2 have: the same length

$n=n_1+n_2$, where n_1 is the number of "0.s" in each of them and n_2 is the number of "1.s" in each of them and the number of agreements between them is $f_1=r_1+r_2$ where r_1 of theme are "0.s" and r_2 of theme are"1.s" such the number of disagreements is $f_2=n-f_1=(n_1-r_1)+(n_2-r_2)=[n-(r_1+r_2)$, by the same way :

If there are two rows from the matrix B, for simplicity B_1, B_2 have: the same length $m=m_1+m_2$, where m_1 is the number of "0.s" in each of them and m_2 is the number of "1.s" in each of them and the number of agreements between them is $g_1=h_1+h_2$ where h_1 of theme are "0.s" and h_2 of theme are"1.s"such the number of disagreements is $g_2=m-g_1=(m_1-h_1)+(m_2-h_2)=[m-(h_1+h)_2]$.

Such the representation of the information as following:

Representation 1: Compare $A_1(B_1)$ and $A_2(B_2)$

$A_1(B_1)\Rightarrow$	$\overbrace{\overbrace{0\ 0\ \dots\ 0}^{r_1}\ \overbrace{0\ 0\ \dots\ 0}^{(n_1-r_1)}}^{n_1\ of\ "0.s"}\ \overbrace{\overbrace{1\ 1\ \dots\ 1}^{(n_2-r_2)}\ \overbrace{1\ 1\ \dots\ 1}^{r_2}}^{n_2\ of\ "1.s""}$ $B_1\ B_1\ \dots\ B_1\ \ B_1\ B_1\ \dots\ B_1\ \ \bar{B}_1\ \bar{B}_1\ \dots\ \bar{B}_1\ \ \bar{B}_1\ \bar{B}_1\ \dots\ \bar{B}_1$
$A_2(B_2)\Rightarrow$	$B_2\ B_2\ \dots\ B_2\ \bar{B}_2\ \bar{B}_2\ \dots\ \bar{B}_2\ \ B_2\ B_2\ \dots\ B_2\ \bar{B}_2\ \bar{B}_2\ \dots\ \bar{B}_2$ $\underbrace{\underbrace{0\ 0\ \dots\ 0}_{r_1}\ \underbrace{1\ 1\ \dots\ 1}_{(n_2-r_2)}}_{n_1 of "0.s"}\ \underbrace{\underbrace{0\ 0\ \dots\ 0}_{(n_1-r_1)}\ \underbrace{1\ 1\ \dots\ 1}_{r_2}}_{n_2 of "1.s"}$

- The number of agreements between A_1 and A_2 is: $f_1=r_1+r_2$.
- The number of disagreements between A_1 and A_2 is: $f_2=n-f_1=[n_1-(r_1+r_2)]$.

  This representation showing a need to compare: B_1 *and* B_2, B_1 *and* $\overline{B}_2$, $\overline{B}_1$ *and* B_2 and $\overline{B}_1$ *and* $\overline{B}_2$,

a. Compare B_1 and B_2 : The Representation of the rows B_1 and B_2 of matrix B is as following:

Representation 2: Compare B_1 and B_2

$B_1\Rightarrow$	$\overbrace{\overbrace{0\ 0\ \dots\ 0}^{h_1}\overbrace{0\ 0\ \dots\ 0}^{(m_1-h_1)}}^{m_1\ of\ "0.s"}\overbrace{\overbrace{1\ 1\ \dots\ 1}^{(m_2-h_2)}\overbrace{1\ 1\ \dots\ 1}^{h_2}}^{m_2\ of\ "1.s"}$
$B_2\Rightarrow$	$\underbrace{\underbrace{0\ 0\ \dots\ 0}_{h_1}\underbrace{1\ 1\ \dots\ 1}_{(m_2-h_2)}}_{m_1 of "0.s"}\ \underbrace{\underbrace{0\ 0\ \dots\ 0}_{(m_1-h_1)}\underbrace{1\ 1\ \dots\ 1}_{h_2}}_{m_2 of "1.s"}$

- The number of agreements between B_1 and B_2 is: $g_1=h_1+h_2$.
- The number of disagreements between B_1 and B_2 is: $g_2=m-g_1=[m-(h_1+h_2)]$.

b. Compare B_1 and $\overline{B}_2$: The Representation of the rows B_1 and $\overline{B}_2$ of matrix B is as following:

Representation 3: Compare B_1 and $\overline{B}_2$

$B_1 \Rightarrow$	$\overbrace{\overbrace{0\ 0\ \dots\ 0}^{h_1}\overbrace{0\ 0\ \dots\ 0}^{(m_1-h_1)}}^{m_1\ of\ "0.s"}\overbrace{\overbrace{1\ 1\ \dots\ 1}^{(m_2-h_2)}\overbrace{1\ 1\ \dots\ 1}^{h_2}}^{m_2\ of\ "1.s"}$
$\overline{B}_2 \Rightarrow$	$\underbrace{\underbrace{1\ 1\ \dots\ 1}_{h_1}\underbrace{0\ 0\ \dots\ 0}_{(m_2-h_2)}}_{m_1 of "1.s"}\ \underbrace{\underbrace{1\ 1\ \dots\ 1}_{(m_1-h_1)}\underbrace{0\ 0\ \dots\ 0}_{h_2}}_{m_2 of "0.s"}$

- The number of agreements between B_1 and $\overline{B}_2$ is: $g_2 = m - g_1 = [m - (h_1 + h_2)]$.
- the number of Disagreements between B_1 and $\overline{B}_2$ is: $g_1 = (h_1 + h_2)$.

c. Compare $\overline{B}_1$ and B_2 : The Representation of the rows B_1 and $\overline{B}_2$ of matrix B is as following:

Representation 4: Compare $\overline{B}_1$ and B_2

$\overline{B}_1 \Rightarrow$	$\overbrace{\overbrace{1\ 1\ \dots\ 1}^{h_1}\overbrace{1\ 1\ \dots\ 1}^{(m_1-h_1)}}^{m_1\ of\ "1.s"}\overbrace{\overbrace{0\ 0\ \dots\ 0}^{(m_2-h_2)}\overbrace{0\ 0\ \dots\ 0}^{h_2}}^{m_2\ of\ "0.s"}$
$B_2 \Rightarrow$	$\underbrace{\underbrace{0\ 0\ \dots\ 0}_{h_1}\underbrace{1\ 1\ \dots\ 1}_{(m_2-h_2)}}_{m_1 of "0.s"}\ \underbrace{\underbrace{0\ 0\ \dots\ 0}_{(m_1-h_1)}\underbrace{1\ 1\ \dots\ 1}_{h_2}}_{m_2 of "1.s"}$

- The number of agreements between $\overline{B}_1$ and B_2 is: $g_2 = m - g_1 = [m - (h_1 + h_2)]$.
- The number of Disagreements between $\overline{B}_1$ and B_2 is: $g_1 = h_1 + h_2$.

d. Compare $\overline{B}_1$ and $\overline{B}_2$: The Representation of the rows $\overline{B}_1$ and $\overline{B}_2$ of matrix B is as following:

Representation 5: Compare $\overline{B}_1$ and $\overline{B}_2$

$\overline{B}_1 \Rightarrow$	$\overbrace{\overbrace{1\ 1\ \dots\ 1}^{h_1}\overbrace{1\ 1\ \dots\ 1}^{(m_1-h_1)}}^{m_1\ of\ "1.s"}\overbrace{\overbrace{0\ 0\ \dots\ 0}^{(m_2-h_2)}\overbrace{0\ 0\ \dots\ 0}^{h_2}}^{m_2\ of\ "0.s"}$
$\overline{B}_2 \Rightarrow$	$\underbrace{\underbrace{1\ 1\ \dots\ 1}_{h_1}\underbrace{0\ 0\ \dots\ 0}_{(m_2-h_2)}}_{m_1 of "1.s"}\ \underbrace{\underbrace{1\ 1\ \dots\ 1}_{(m_1-h_1)}\underbrace{0\ 0\ \dots\ 0}_{h_2}}_{m_2 of "0.s"}$

- The number of agreements between $\overline{B}_1$ and $\overline{B}_2$ is; $g_1 = h_1 + h_2$.
- The number of disagreements between $\overline{B}_1$ and $\overline{B}_2$ is: $g_2 = [m - (h_1 + h_2)]$.

First Step: For one row A_i of the matrix *A* and two different rows B_j, B_k of the matrix *B* then from the representations (2,5) and from the following representation of A_i, B_j, B_k:

Representation 6: Compare $A_i(B_j)$ and $A_i(B_k)$

$A_i(B_j) \Rightarrow$	$\overbrace{\overbrace{0\;0\;\dots\;0}^{r_1}\;\overbrace{0\;0\;\dots\;0}^{(n_1-r_1)}}^{n_1\ of\ "0.s"}\;\overbrace{\overbrace{1\;1\;\dots\;1}^{(n_2-r_2)}\;\overbrace{1\;1\;\dots\;1}^{r_2}}^{n_2\ of\ "1.s""}$ $B_j\,B_j\,\dots\,B_j\;\;B_j\,B_j\,\dots\,B_j\;\;\bar{B}_j\,\bar{B}_j\,\dots\,\bar{B}_j\;\;\bar{B}_j\,\bar{B}_j\,\dots\,\bar{B}_j$
$A_i(B_k) \Rightarrow$	$B_k\,B_k\,\dots\,B_k\;B_k\,B_k\,\dots\,B_k\;\;\bar{B}_k\,\bar{B}_k\,\dots\,\bar{B}_k\;\bar{B}_k\,\bar{B}_k\,\dots\,\bar{B}_k$ $\underbrace{\underbrace{0\;0\;\dots\;0}_{r_1}\;\underbrace{0\;0\;\dots\;0}_{(n_1-r_1)}}_{n_1 of "0.s"}\;\underbrace{\underbrace{1\;1\;\dots\;1}_{(n_2-r_2)}\;\underbrace{1\;1\;\dots\;1}_{r_2}}_{n_2 of "1.s"}$

We have :

- The length of $A_i(B_j)$ and $A_i(B_k)$ is: *n.m* .
- The number of "0.s" in each of them is: $n_1m_1 + n_2m_2$.
- The number of "1.s" in each of them is: $n_1m_2 + n_2m_1$.
- The number of agreements is: $n\,g_1 = n(h_1 + h_2)$.
- The number of disagreements is: $n\,g_2 = n[m-(h_1+h_2)]$.
- The difference d between the disagreements and agreements is:

$$d = \left|n\,g_2 - n\,g_1\right| = \left|n[m-(h_1+h_2)] - n(h_1+h_2)\right|$$

$$d = \left|n[m-2(h_1+h_2)]\right|$$

Second Step: For two different rows A_i, A_j of the matrix *A* and one row B_k of the matrix *B* and from the representations (2-5) and from the following representation of A_i, A_j, B_k:

Representation 7: Compare $A_i(B_k)$ and $A_j(B_k)$

$A_i(B_k) \Rightarrow$	$\overbrace{\overbrace{0\;0\;\dots\;0}^{r_1}\;\overbrace{0\;0\;\dots\;0}^{(n_1-r_1)}}^{n_1\ of\ "0.s"}\;\overbrace{\overbrace{1\;1\;\dots\;1}^{(n_2-r_2)}\;\overbrace{1\;1\;\dots\;1}^{r_2}}^{n_2\ of\ "1.s""}$ $B_k\,B_k\,\dots\,B_k\;\;B_k\,B_k\,\dots\,B_k\;\;\bar{B}_k\,\bar{B}_k\,\dots\,\bar{B}_k\;\;\bar{B}_k\,\bar{B}_k\,\dots\,\bar{B}_k$
$A_j(B_k) \Rightarrow$	$B_k\,B_k\,\dots\,B_k\;\bar{B}_k\,\bar{B}_k\,\dots\,\bar{B}_k\;\;B_k\,B_k\,\dots\,B_k\;\bar{B}_k\,\bar{B}_k\,\dots\,\bar{B}_k$ $\underbrace{\underbrace{0\;0\;\dots\;0}_{r_1}\;\underbrace{1\;1\;\dots\;1}_{(n_2-r_2)}}_{n_1 of "0.s"}\;\underbrace{\underbrace{0\;0\;\dots\;0}_{(n_1-r_1)}\;\underbrace{1\;1\;\dots\;1}_{r_2}}_{n_2 of "1.s"}$

We have :

- The length of $A_i(B_k)$ and $A_j(B_k)$ is: *n.m* .
- The number of "0.s" in each of them is: $n_1m_1 + n_2m_2$.
- The number of "1.s" in each of them is: $n_1m_2 + n_2m_1$.
- The number of agreements is: $m\,f_1 = m(r_1 + r_2)$.
- The number of disagreements is: $m\,f_2 = m[n-(r_1+r_2)]$.
- The difference d between the disagreements and agreements is:

$$d = \left|m\,f_2 - m\,f_1\right|\left|m[n-(r_1+r_2)] - m(r_1+r_2)m\right|$$

$$d = \left|[n-2(r_1+r_2)]m\right|$$

Third Step: For two different rows A_i, A_j of the matrix *A* and two different rows B_k, B_l of the Matrix *B* and from the representations (1-5) and from the following representation of A_i, A_j , B_k, B_l :

Representation 8: Compare $A_i(B_k)$ and $A_j(B_l)$

$A_i(B_k) \Rightarrow$	$\begin{array}{c} \overbrace{\overbrace{0\ 0\ \dots\ 0}^{r_1}\ \overbrace{0\ 0\ \dots\ 0}^{(n_1-r_1)}}^{n_1\ of\ "0.s"}\ \overbrace{\overbrace{1\ 1\ \dots\ 1}^{(n_2-r_2)}\ \overbrace{1\ 1\ \dots\ 1}^{r_2}}^{n_2\ of\ "1.s""} \\ B_k\ B_k \dots B_k\ \ B_k\ B_k \dots B_k\ \ \bar{B}_k\ \bar{B}_k \dots \bar{B}_k\ \ \bar{B}_k\bar{B}_k \dots \bar{B}_k \end{array}$
$A_j(B_l) \Rightarrow$	$\begin{array}{c} B_l\ B_l \dots B_l\ \bar{B}_l\ \bar{B}_l \dots \bar{B}_l\ \ B_l\ B_l \dots B_l\ \bar{B}_l\ \bar{B}_l \dots \bar{B}_l \\ \underbrace{\underbrace{0\ 0\ \dots\ 0}_{r_1}\ \underbrace{1\ 1\ \dots\ 1}_{(n_2-r_2)}}_{n_1 of "0.s"}\ \underbrace{\underbrace{0\ 0\ \dots\ 0}_{(n_1-r_1)}\ \underbrace{1\ 1\ \dots\ 1}_{r_2}}_{n_2 of "1.s"} \end{array}$

We have:

- The number of agreements between A_i and A_j is: $f_1 = (r_1 + r_2)$.
- The number of disagreements between A_i and A_j is: $f_2 = [n - (r_1 + r_2)]$
- The number of agreements between B_k and B_l is: $g_1 = (h_1 + h_2)$.
- The number of disagreements between B_k and B_l is: $g_2 = [m - (h_1 + h_2)]$
- The length of $A_i(B_k)$ and $A_j(B_l)$ is: *n.m* .
- The number of “0.s” in each of them is: $n_1m_1 + n_2m_2$.
- The number of “1.s” in each of them is: $n_1m_2 + n_2m_1$.
- The number of agreements between $A_i(B_k)$ and $A_j(B_l)$ is: $f_1g_1 + f_2g_2$, or is: $(r_1 + r_2)(h_1 + h_2) + [n - (r_1 + r_2)][m - (h_1 + h_2)]$
- The number of disagreements between $A_i(B_k)$ and $A_j(B_l)$ is: $f_1g_2 + f_2g_1$, or is: $(r_1 + r_2)[m - (h_1 + h_2)] + [n - (r_1 + r_2)](h_1 + h_2)$
- The difference *d* between the agreements and disagreements of $A_i(B_k)$ and $A_j(B_l)$ is:

$$d = \left|(f_1g_1 + f_2g_2) - (f_1g_2 + f_2g_1)\right|$$

$$d = \left|(f_1 - f_2)(g_1 - g_2)\right|$$

Or:

$$d = nm - 2n(h_1 + h_2) - 2m(r_1 + r_2) + 4(r_1 + r_2)(h_1 + h_2)$$

Example 4. Using binary representations of the standard Hadmard matrices:

$$H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, H_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, ,...H_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, H_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \cdots, H_{4h_1}, ...$$

The set of row of a standard Hadamard matrix form an additive group for addition by *mod* 2 and the set of this rows except the first row also closed under the addition (Walsh's Sequences).

Except H_2 , First row contains only "0.s" each of the other rows contains $2h_1$ of "0.s" , $2h_1$ of "1.s", $2h_1$ of disagreements and $2h_1$ of agreements, h_1 of the agreements are "0.s" and the other h_1 are "1.s".

i. Is very clear: H2 (H2) = H4 is an Hadamard matrix.

ii. All entries of the first row in $H_2(H_{4h})$ are "0" and each other row contains 4*h* of "0.s" and 4*h* of "1.s" and orthogonal with the first row, the order of the matrix $H_2(H_{4h})$ is 8*h*.

$$H_2(H_{4h}) = \begin{bmatrix} H_{4h} & H_{4h} \\ H_{4h} & \overline{H_{4h}} \end{bmatrix}$$

iii. Each row except the first row contains 4*h* of "0.s", 4*h* of "1.s" and any two different rows except the first contain 4*h* of disagreements and 4*h* of agreements (from *third.* The difference between the agreements and disagreements is: $d = (f_1 - f_2)(g_1 - g_2) = (1-1)(2h-2h) = 0$) 2*h* of the agreements are "0.s" and the other 2*h* are "1.s" also because this set of rows is closed under the addition. Thus these two rows are orthogonal and the columns of $H_2(H_{4h})$ have the same properties of the rows and $H_2(H_{4h})$ is an Hadamard matrix. By the same way for $H_{4h}(H_2)$ is an Hadamard matrix.

Example 5.

$$H_2(H_{12}) = \left[\begin{array}{cccccccccccc|cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&1&1&0&1&1&0&0&0&0&1&1&0&1&1&0&1&1&0\\
0&0&0&1&0&1&1&0&1&1&0&1&0&0&0&1&0&1&1&0&1&1&0&1\\
0&1&1&1&0&0&0&0&0&1&1&1&0&1&1&1&0&0&0&0&0&1&1&1\\
0&1&0&0&1&1&0&0&1&1&1&0&0&1&0&0&1&1&0&0&1&1&1&0\\
0&0&1&0&1&1&0&1&0&1&0&1&0&0&1&0&1&1&0&1&0&1&0&1\\
0&1&1&1&1&1&1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0&0\\
0&1&0&1&1&0&0&1&1&0&0&1&0&1&0&1&1&0&0&1&1&0&0&1\\
0&0&1&1&0&1&0&1&1&0&1&0&0&0&1&1&0&1&0&1&1&0&1&0\\
0&1&1&0&0&0&1&1&1&1&0&0&0&1&1&0&0&0&1&1&1&1&0&0\\
0&1&0&0&0&1&1&1&0&0&1&1&0&1&0&0&0&1&1&1&0&0&1&1\\
0&0&1&0&1&0&1&0&1&0&1&1&0&0&1&0&1&0&1&0&1&0&1&1\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&1&0&1&1&0&1&1&0&1&1&1&0&0&1&0&0&1&0&0&1\\
0&0&0&1&0&1&1&0&1&1&0&1&1&1&1&0&1&0&0&1&0&0&1&0\\
0&1&1&1&0&0&0&0&0&1&1&1&1&0&0&0&1&1&1&1&1&0&0&0\\
0&1&0&0&1&1&0&0&1&1&1&0&1&0&1&1&0&0&1&1&0&0&0&1\\
0&0&1&0&1&1&0&1&0&1&0&1&1&1&0&1&0&0&1&0&1&0&1&0\\
0&1&1&1&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&1&1\\
0&1&0&1&1&0&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&1&0\\
0&0&1&1&0&1&0&1&1&0&1&0&1&1&0&0&1&0&1&0&0&1&0&1\\
0&1&1&0&0&0&1&1&1&1&0&0&1&0&0&1&1&1&0&0&0&0&1&1\\
0&1&0&0&0&1&1&1&0&0&1&1&1&0&1&1&1&0&0&0&1&1&0&0\\
0&0&1&0&1&0&1&0&1&0&1&1&1&1&0&1&0&1&0&1&0&1&0&0
\end{array}\right]$$

In this matrix, except the first row (column), each row (column) contains 12 0f “0.s” , 12 of “1.s”, 12 of disagreements and 12 of agreements, 6 of the agreements are “0” and 6 of agreements are “1.s”

iv. All entries of the first row in $H_{4h_1}(H_{4h2})$ are “0” and each other row contains $8h_1h_2$ of “0.s” and $8h_1h_2$ of “1.s” and orthogonal with the first row and the order of the matrix is: $16h_1h_2$.

For any two different rows, except the first row

- The number of agreements between A_i and A_j is: $f_1 = (h_1 + h_1) = 2h_1$.
- The number of disagreements between A_i and A_j is: $f_2 = [4h_1 - 2h_1] = 2h_1$
- The number of agreements between B_k and B_l is: $g_1 = (h_2 + h_2) = 2h_2$.
- The number of disagreements between B_k and B_l is: $g_2 = [4h_2 - 2h_2] = 2h_2$

❖ In the composition $A_i(B_j)$ and $A_i(B_k)$:

- The number of agreements is: $n\,g_1 = 4h_1(2h_2) = 8h_1h_2$.
- The number of disagreements is: $n\,g_2 = 4h_1(2h_2) = 8h_1h_2$.
- The difference d between the disagreements and agreements is: $d = 0$

Thus, second condition of the orthogonal is verified.

❖ In the composition $A_i(B_k)$ and $A_j(B_k)$:

- The number of agreements is: $m\,f_1 = 4h_2(2h_1) = 8h_1h_2$.
- The number of disagreements is: $m\,f_2 = 4h_2(2h_1)$.
- The difference d between the disagreements and agreements is: $d = 0$

Thus, second condition of the orthogonal is verified.

❖ In the composition $A_i(B_k)$ and $A_j(B_l)$:

- The number of agreements is: $f_1g_1 + f_2g_2 = 2h_1(2h_2) + 2h_1(2h_2) = 8h_1h_2$.
- The number of disagreements is: $f_1g_2 + f_2g_1 = 2h_1(2h_2) + 2h_1(2h_2) = 8h_1h_2$.
- The difference d between the disagreements and agreements is “0”, or :

$$d = |(2h_1 - 2h_1)(2h_2 - 2h_2)| = 0$$

- The columns of $H_{4h_1}(H_{4h2})$ have the same properties of the rows and $H_{4h_1}(H_{4h2})$ is an Hadamard matrix. By the same way for $H_{4h_2}(H_{4h_1})$ is an Hadamard matrix.

Thus, second condition of the orthogonal is verified and rows of *A*(*B*) except the first row form an orthogonal set

Second. Compose binary matrices is associative: We will prove that $A(B(C)) = (A(B))(C)$ for any *A*, *B*, *C* binary matrices.

Suppose:

$$A = [a_{ij}]_{h_1} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_{h_1} \end{bmatrix}, B = [b_{kl}]_{h_2} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{h_2} \end{bmatrix}, C = [c_{mn}]_{h_3} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{h_3} \end{bmatrix}$$

From the left side:

i. $A(B(C)) = \lfloor a_{ij}(B(C)) \rfloor_{h_1.h_2.h_3},\ i, j = 1,2,...,h_1$

- If $a_{ij} = 0$ then:

From the left side:

$$a_{ij}(B(C)) = (B(C)) = \begin{bmatrix} B_1(C) \\ B_2(C) \\ \cdots \\ B_{h_2}(C) \end{bmatrix} = \begin{bmatrix} b_{11}(C) & b_{12}(C) & .... & b_{1h_2}(C) \\ b_{21}(C) & b_{22}(C) & .... & b_{2h_2}(C) \\ .... & .... & .... & .... \\ b_{h_21}(C) & b_{h_22}(C) & .... & b_{h_21h_2}(C) \end{bmatrix}$$

From the right side:

$$(a_{ij}(B))(C) = (B)(C) = \begin{bmatrix} B_1(C) \\ B_2(C) \\ \cdots \\ B_{h_2}(C) \end{bmatrix} = \begin{bmatrix} b_{11}(C) & b_{12}(C) & .... & b_{1h_2}(C) \\ b_{21}(C) & b_{22}(C) & .... & b_{2h_2}(C) \\ .... & .... & .... & .... \\ b_{h_21}(C) & b_{h_22}(C) & .... & b_{h_21h_2}(C) \end{bmatrix}$$

Thus: *A*(*B*(*C*)) = (*A*(*B*)(*C*)

- If $a_{ij} = 1$ then:

From the left side:

$$a_{ij}(B(C)) = \overline{(B(C))} = \begin{bmatrix} \overline{B_1(C)} \\ \overline{B_2(C)} \\ \cdots \\ \overline{B_{h_2}(C)} \end{bmatrix} = \begin{bmatrix} \overline{b_{11}(C)} & \overline{b_{12}(C)} & \cdots\cdots & \overline{b_{1h_2}(C)} \\ \overline{b_{21}(C)} & \overline{b_{22}(C)} & \cdots\cdots & \overline{b_{2h_2}(C)} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \overline{b_{h_21}(C)} & \overline{b_{h_22}(C)} & \cdots & \overline{b_{h_21h_2}(C)} \end{bmatrix}$$

From the right side:

$$(a_{ij}(B))(C) = (\bar{B}(C)) = \begin{bmatrix} \bar{B}_1(C) \\ \bar{B}_2(C) \\ \cdots \\ \bar{B}_{h_2}(C) \end{bmatrix} = \begin{bmatrix} \overline{b_{11}}(C) & \overline{b_{12}}(C) & \cdots\cdots & \overline{b_{1h_2}}(C) \\ \overline{b_{21}}(C) & \overline{b_{22}}(C) & \cdots\cdots & \overline{b_{2h_2}}(C) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \overline{b_{h_21}}(C) & \overline{b_{h_22}}(C) & \cdots & \overline{b_{h_21h_2}}(C) \end{bmatrix}$$

Is $\bar{B}(C) = \overline{B(C)}$?

- If $b_{kl} = 0 \Rightarrow \begin{cases} \overline{b_{kl}}(C) = \bar{0}(C) = \bar{C} \\ \overline{b_{kl}(C)} = \overline{0(C)} = \bar{C} \end{cases}$, if $b_{kl} = 1 \Rightarrow$

$$\begin{cases} \overline{b_{kl}}(C) = \bar{1}(C) = 0(C) = C \\ \overline{b_{kl}(C)} = \overline{1(C)} = \bar{\bar{C}} = C \end{cases}$$

Thus $\bar{B}(C) = \overline{B(C)}$

Thus for any binary matrices *A*, *B*, *C* : *A*(*B*(*C*)) = (*A*(*B*)(*C*) and compose binary matrices is associative.

Result: If $n = n_1 n_2 n_3$ and there are Hadmard matrices of orders n_1 , n_2 , n_3 then there is Hadmard matrix of order $n = n_1 n_2 n_3$.

Third. There is Identity for compose binary matrices that is Null matrix *O* = [0] and for any binary matrix *A* is *A*(*O*) = *O*(*A*) = *A*.

Forth. Compose binary matrices is not commutative:

Example 6:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

And clear that:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) \neq \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

Fifth. For the binary matrix *A* there is no inverse *A*' such *A*(*A*') = *A*'(*A*) = *O* = [0].

Example 7: Suppose $(a) = a_0, a_1, a_2, ..., a_{n-1}$ and $(b) = b_0, b_1, b_2, ..., b_{m-1}$ are one period of two binary M-Sequences with the lengths $n = 2^{k_1} - 1$, $m = 2^{k_2} - 1$ respectively, and the matrix *A* is the all cyclic permutations of the sequence (*a*), the matrix *B* is the all cyclic permutation of the matrix (b).

Thus each row of the matrix *A* contains $n_1 = (2^{k_1-1} - 1)$ of "0.s" and $n_2 = 2^{k_1-1}$ of "1.s".

The rows of the matrix *A* are closed under the addition *mod* 2 and any two different rows are orthogonal and contain $f_2 = 2^{k_1-1}$ disagreements and $f_1 = (2^{k_1-1} - 1)$ of agreements, $r_1 = 2^{k_1-2}$ of agreements are "1.s" and $r_2 = (2^{k_1-2} - 1)$ of agreements are "0.s".

Thus each row of the matrix *B* contains $m_1 = (2^{k_2-1} - 1)$ of "0.s" and $m_2 = 2^{k_2-1}$ of "1.s".

The rows of the matrix *B* are closed under the addition *mod* 2 and any two different rows are orthogonal and contain $g_2 = 2^{k_2-1}$ disagreements and $g_1 = (2^{k_2-1} - 1)$ of agreements, $h_1 = 2^{k_2-2}$ of agreements are "1.s" and $h_2 = (2^{k_2-2} - 1)$ of agreements are "0.s".

❖ In the composition *A*(*B*):

- The length of each row is: $nm=(2^{k_1}-1)(2^{k_2}-1)=2^{k_1+k_2}-(2^{k_1}+2^{k_2})+1$
- The number of "1.s" is: $n_1m_1+n_2m_2=(2^{k_1-1}-1)(2^{k_2-1}-1)+2^{k_1-1}2^{k_2-1}$

$n_1m_1+n_2m_2=2^{k_1+k_2-1}-(2^{k_1-1}+2^{k_2-1})+1$

- The number of "0.s" is: $n_1m_2+n_2m_1=(2^{k_1-1}-1)2^{k_2-1}+2^{k_1-1}(2^{k_2-1}-1)$
  $n_1m_2+n_2m_1=2^{k_1+k_2-1}-(2^{k_1-1}+2^{k_2-1})$
- The difference between the number of "0.s" and the number of "1.s" is 1. Thus, the first condition of the orthogonal is exist.

❖ In the composition $A_i(B_j)$ and $A_i(B_k)$:

- The number of agreements is: $ng_1=(2^{k_1}-1)(2^{k_2-1}-1)=2^{k_1+k_2-1}-(2^{k_1}+2^{k_2-1})+1$
- The number of disagreements is: $ng_2=(2^{k_1}-1)(2^{k_2-1})=2^{k_1+k_2-1}-2^{k_2-1}$.
- The difference *d* between the disagreements and agreements is: $d=2^{k_1}-1$ Thus, second condition of the orthogonal is unverified.

❖ In the composition $A_i(B_k)$ and $A_j(B_k)$:

- The number of agreements is: $mf_1=(2^{k_2}-1)(2^{k_1-1}-1)=2^{k_1+k_2-1}-(2^{k_1-1}+2^{k_2})+1$.
- The number of disagreements is: $mf_2=(2^{k_2}-1)(2^{k_1-1})=2^{k_1+k_2-1}-2^{k_1-1}$.
- The difference *d* between the disagreements and agreements is: $d=2^{k_2}-1$ Thus, second condition of the orthogonal is unverified.

❖ In the composition $A_i(B_k)$ and $A_j(B_l)$:

- The number of agreements is: $f_1g_1+f_2g_2$.
- The number of disagreements is: $f_1g_2+f_2g_1$.
- The difference *d* between the disagreements and agreements is:

$$d=\left|(f_1-f_2)(g_1-g_2)\right|=\left|[(2^{k_1-1}-1)-2^{k_1-1}][(2^{k_2-1}-1)-2^{k_2-1}]\right|=1$$

Thus, second condition of the orthogonal is verified

For remove the non-orthogonal situations need think about extend the rows of the matrix *A* and the rows of the matrix *B* by adding "0" at the beginning each row of them, thus we have:

$n=2^{k_1},\ n_1=2^{k_1-1},\ n_2=2^{k_1-1},\ f_1=f_2=2^{k_1-1}$

$m=2^{k_2},\ m_1=2^{k_2-1},\ m_2=2^{k_2-1},\ g_1=g_2=2^{k_2-1}$

❖ In the composition $\tilde{A}(\tilde{B})$: where the "~" is the signal of extending, and $\tilde{A}(\tilde{B})$ of the size $(2^{k_1}-1)(2^{k_2}-1)\times 2^{k_1}2^{k_2}$.

- The length of each row is: $nm=2^{k_1}2^{k_2}=2^{k_1+k_2}$
- The number of "1.s" is: $n_1m_1+n_2m_2=2^{k_1-1}2^{k_2-1}+2^{k_1-1}2^{k_2-1}=2^{k_1+k_2-1}$
- The number of "0.s" is: $n_1m_2+n_2m_1=2^{k_1-1}2^{k_2-1}+2^{k_1-1}2^{k_2-1}=2^{k_1+k_2-1}$
- The difference between the number of "0.s" and the number of "1.s" is 0.

Thus, the first condition of the orthogonal is verified.

❖ In the composition $\tilde{A}_i(\tilde{B}_j)$ and $\tilde{A}_i(\tilde{B}_k)$:

- The number of agreements is: $ng_1=2^{k_1}2^{k_2-1}=2^{k_1+k_2-1}$.
- The number of disagreements is: $ng_2=2^{k_1}2^{k_2-1}=2^{k_1+k_2-1}$.
- The difference *d* between the disagreements and agreements is: $d=0$

Thus, second condition of the orthogonal is verified.

❖ In the composition $\tilde{A}_i(\tilde{B}_k)$ and $\tilde{A}_j(\tilde{B}_k)$:

- The number of agreements is: $mf_1=2^{k_2}2^{k_1-1}=2^{k_1+k_2-1}$.
- The number of disagreements is: $mf_2=2^{k_2}2^{k_1-1}=2^{k_1+k_2-1}$.
- The difference *d* between the disagreements and agreements is: $d=0$

Thus, second condition of the orthogonal is verified.

❖ In the composition $\tilde{A}_i(\tilde{B}_k)$ and $\tilde{A}_j(\tilde{B}_l)$:

- The number of agreements is: $f_1g_1 + f_2g_2$.
- The number of disagreements is: $f_1g_2 + f_2g_1$.
- The difference d between the disagreements and agreements is: $d = |(f_1 - f_2)(g_1 - g_2)| = 0$

Thus, second condition of the orthogonal is verified.

From example 1. And example 2.:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

And

$$A(B) = \left[\begin{array}{ccccccc|ccccccc|ccccccc}
0&1&1&0&1&0&0&1&0&0&1&0&1&1&0&1&1&0&1&0&0\\
0&0&1&1&0&1&0&1&1&0&0&1&0&1&0&0&1&1&0&1&0\\
0&0&0&1&1&0&1&1&1&1&0&0&1&0&0&0&0&1&1&0&1\\
1&0&0&0&1&1&0&0&1&1&1&0&0&1&1&0&0&0&1&1&0\\
0&1&0&0&0&1&1&1&0&1&1&1&0&0&0&1&0&0&0&1&1\\
1&0&1&0&0&0&1&0&1&0&1&1&1&0&1&0&1&0&0&0&1\\
1&1&0&1&0&0&0&0&0&1&0&1&1&1&1&1&0&1&0&0&0\\
\hline
0&1&1&0&1&0&0&0&1&1&0&1&0&0&1&0&0&1&0&1&1\\
0&0&1&1&0&1&0&0&0&1&1&0&1&0&1&1&0&0&1&0&1\\
0&0&0&1&1&0&1&0&0&0&1&1&0&1&1&1&1&0&0&1&0\\
1&0&0&0&1&1&0&1&0&0&0&1&1&0&0&1&1&1&0&0&1\\
0&1&0&0&0&1&1&0&1&0&0&0&1&1&1&0&1&1&1&0&0\\
1&0&1&0&0&0&1&1&0&1&0&0&0&1&0&1&0&1&1&1&0\\
1&1&0&1&0&0&0&1&1&0&1&0&0&0&0&0&1&0&1&1&1\\
\hline
1&0&0&1&0&1&1&0&1&1&0&1&0&0&0&1&1&0&1&0&0\\
1&1&0&0&1&0&1&0&0&1&1&0&1&0&0&0&1&1&0&1&0\\
1&1&1&0&0&1&0&0&0&0&1&1&0&1&0&0&0&1&1&0&1\\
0&1&1&1&0&0&1&1&0&0&0&1&1&0&1&0&0&0&1&1&0\\
1&0&1&1&1&0&0&0&1&0&0&0&1&1&0&1&0&0&0&1&1\\
0&1&0&1&1&1&0&1&0&1&0&0&0&1&1&0&1&0&0&0&1\\
0&0&1&0&1&1&1&1&1&0&1&0&0&0&1&1&0&1&0&0&0
\end{array}\right]$$

❖ In the composition $A(B)$:

- The length of each row is: $nm = (2^{k_1} - 1)(2^{k_2} - 1) = 3(7) = 21$
- The number of “1.s” is: $n_1m_1 + n_2m_2 = 1(3) + 2(4) = 11$
- The number of “0.s” is: $n_1m_2 + n_2m_1 = 1(4) + 2(3) = 10$
- The difference between the number of “0.s” and the number of “1.s” is 1.

Thus, the first condition of the orthogonal is exist.

❖ In the composition $A_i(B_j)$ and $A_i(B_k)$:

- The number of agreements is: $n\,g_1 = 3(3) = 9$
- The number of disagreements is: $n\,g_2 = 3(4) = 12$.
- The difference d between the disagreements and agreements is: $d = 3$.

For example, the first seven rows, or between 8-14 rows, or between 15-21 rows. Thus, second condition of the orthogonal is unverified.

❖ In the composition $A_i(B_k)$ and $A_j(B_k)$:

- The number of agreements is: $m\, f_1 = 7(1) = 7$.
- The number of disagreements is: $m\, f_2 = 7(2) = 14$.
- The difference *d* between the disagreements and agreements is: $d = 7$.

For examples: between the 1th row and 8th row, between 2th row and 9th row, …Thus, second condition of the orthogonal is unverified.

❖ In the composition $A_i(B_k)$ and $A_j(B_l)$:

- The number of agreements is: $f_1 g_1 + f_2 g_2 = 1(3) + 2(4) = 11$.
- The number of disagreements is: $f_1 g_2 + f_2 g_1 = 1(4) + 2(3) = 10$.
- The difference *d* between the disagreements and agreements is:

$$d = \left|(f_1 - f_2)(g_1 - g_2)\right| = \left|[(2^{k_1-1} - 1) - 2^{k_1-1}][(2^{k_2-1} - 1) - 2^{k_2-1}]\right| = 1$$

For example: between the 1th and 9th row, between 1th row and 10throw, …, second condition of the orthogonal is verified.

By extend the matrices *A* and *B*:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \tilde{\tilde{B}} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{A}(\tilde{B}) = \left[\begin{array}{cccccccccccccccccccccccccccccccc}
0&1&0&0&1&0&1&1&1&0&1&1&0&1&0&0&0&1&0&0&1&0&1&1&1&0&1&1&0&1&0&0 \\
0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0&0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0 \\
0&1&1&1&0&0&1&0&1&0&0&0&1&1&0&1&0&1&1&1&0&0&1&0&1&0&0&0&1&1&0&1 \\
0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&0&0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&0 \\
0&1&0&1&1&1&0&0&1&0&1&0&0&0&1&1&0&1&0&1&1&1&0&0&1&0&1&0&0&0&1&1 \\
0&0&1&0&1&1&1&0&1&1&0&1&0&0&0&1&0&0&1&0&1&1&1&0&1&1&0&1&0&0&0&1 \\
0&0&0&1&0&1&1&1&1&1&1&0&1&0&0&0&0&0&0&1&0&1&1&1&1&1&1&0&1&0&0&0 \\
0&1&0&0&1&0&1&1&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&0&1&0&0&1&0&1&1 \\
0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0&1&0&0&1&1&0&1&0&0&1&1&0&0&1&0&1 \\
0&1&1&1&0&0&1&0&1&0&0&0&1&1&0&1&1&0&0&0&1&1&0&1&0&1&1&1&0&0&1&0 \\
0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&0&1&1&0&0&0&1&1&0&0&0&1&1&1&0&0&1 \\
0&1&0&1&1&1&0&0&1&0&1&0&0&0&1&1&1&0&1&0&0&0&1&1&0&1&0&1&1&1&0&0 \\
0&0&1&0&1&1&1&0&1&1&0&1&0&0&0&1&1&1&0&1&0&0&0&1&0&0&1&0&1&1&1&0 \\
0&0&0&1&0&1&1&1&1&1&1&0&1&0&0&0&1&1&1&0&1&0&0&0&0&0&0&1&0&1&1&1 \\
0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0 \\
0&1&1&0&0&1&0&1&0&1&1&0&0&1&0&1&1&0&0&1&1&0&1&0&1&0&0&1&1&0&1&0 \\
0&1&1&1&0&0&1&0&0&1&1&1&0&0&1&0&1&0&0&0&1&1&0&1&1&0&0&0&1&1&0&1 \\
0&0&1&1&1&0&0&1&0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&0&1&1&0&0&0&1&1&0 \\
0&1&0&1&1&1&0&0&0&1&0&1&1&1&0&0&1&0&1&0&0&0&1&1&1&0&1&0&0&0&1&1 \\
0&0&1&0&1&1&1&0&0&0&1&0&1&1&1&0&1&1&0&1&0&0&0&1&1&1&0&1&0&0&0&1 \\
0&0&0&1&0&1&1&1&0&0&0&1&0&1&1&1&1&1&1&0&1&0&0&0&1&1&1&0&1&0&0&0
\end{array}\right]$$

The length of any row is: 4(8) = 32 and contains 16 of “0.s” and 16 of “1.s”. Any two different row contain 16 of agreements and 16 of disagreements and rows of $\tilde{A}(\tilde{B})$ is closed under the addition by *mod* 2. Thus rows of $\tilde{A}(\tilde{B})$ is an orthogonal set. Coming back to system {1,-1} replacing each “1” by “-1” and each “0” by “1”.

Conclusion

1. The operation of compose binary matrices is associative, non-commutative, there is identity and no inverse.

2. compose two Hadamard matrices is also Hadamard matrix.

3. If $n = n_1.n_2.....n_k$ and there are Hadamard matrices of orders n_1, n_2, …, n_k then there is Hadamard matrix of order n

4. Using compose matrices we can get Hadamard matrices with the bigger lengths and the bigger minimum distance that assists to increase secrecy of these information and increase the possibility of correcting mistakes resulting in the channels of communication.

5. Compose two matrices of M-Sequences is not matrix of M-Sequences and the result of composition is not orthogonal set.

6. we can extend the matrices of M-Sequences to getting orthogonal matrix with big size, with the bigger lengths and the bigger minimum distance that assists to increase secrecy of these information and increase the possibility of correcting mistakes resulting in the channels of communication.

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