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Pyramid Power in Colors

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ABSTRACT

The infamous ‘pyramid’ question on the Math section of the Preliminary Scholastic Aptitude Test (PSAT) was reconsidered. Illustrations in colors are presented with a new possible scoring key. Some practical consequential issues are discussed.

Keywords: item analysis, scoring errors, test validity.

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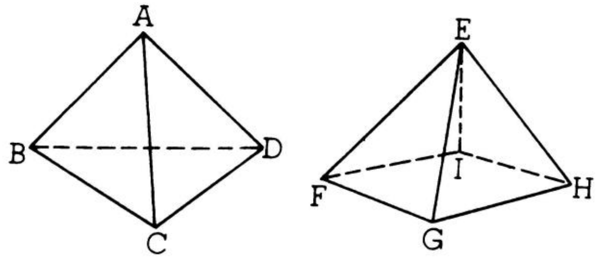
Introduction

This manuscript tries to address unsolved issues for the pyramid question first appeared in Wainer (1981). Wainer (1981, 1983) published a thought provoking paper that illustrated the importance of preparing sound scoring keys for a standardized test. Wainer (1981) also presented the basic concepts related to both classical and model-based item analysis and discussed the practically important concepts including consequential validity. Although this manuscript contains some mathematics concepts in its presentation, it is fundamentally relevant to educational assessment. Initially from the secondary source (i.e., Bock et al. 1991) which still remains as an unpublished work, the leading author was first exposed to the pyramid question (see also Thissen and Orlando 2001; Wainer 1989).

Here is the question as it appeared on the test from Wainer (1981):

The Question:



44. In pyramids ABCD and EFGHI shown above, all faces except base FGHI are equilateral triangles of equal size. If face ABC were placed on face EFG so that the vertices of the triangles coincide, how many exposed faces would the resulting solid have?

(A) Five (B) Six (C) Seven
(D) Eight (E) Nine

Figure 1: The pyramid question as it appeared on the PSAT.

It is a geometric question with the original intended key ‘(C) Seven’ to be the correct choice. The initial key was later found out to be incorrect and another key ‘(A) Five’ emerged to be the correct choice. The Educational Testing Service (ETS) and the College Board ultimately scored both answers as correct after having found out another key. As indicated in Fiske (1981) the question had appeared on old forms of the Scholastic Aptitude Test (SAT; now the same acronym stands for Scholastic Assessment Test). We are presenting an argument in this manuscript that another answer to the question may be ‘(D) Eight’ without any modifications to the original question. In our research on this issue, we have not found any suggestions related to our argument although the same answer was suggested by a Harvard graduate student, Lawrence A. Denenberg (see Student Finds Third PSAT Answser, 1981).

Before presenting our argument, let’s examine the empirical results from item analysis. The numbers we are reporting are directly obtained from Wainer (1983) and Oderwald (1983). Wainer (1983) wrote a fascinating, instructional work and presented all the raw numbers from which the numbers in Table 1 can certainly be reconstructed, if required. There were 50 questions on the Math section of the PSAT, and the total score in Table 1 was based on the summed score of the remaining dichotomously scored 49 items.

As indicated in Wainer (1983) and Oderwald (1983), the initial key acted exactly like the usual correct option in multiple choice questions. Both the initial key and the new key possess positive values of point-biserial correlation (i.e., Pearson product moment correlation between a binary indicator variable and a quantitative variable) implying that more able examinees tended to choose these two options. This might have been the justification of the ETS and the College Board's decision to treat both as correct. The values of the mean total score from the examinees for the two options were higher than the rest of the groups. Illustrations from Wainer's (1983) are based on conditional, empirical density functions and empirical trace lines of the conditional responses and, hence, more exciting and formative than these oversimplified summary statistics based on item analysis from the Pearsonian classical test theory framework. Interested readers are encouraged to read Wainer (1983) for more detailed statistical procedures of item analysis.

Table 1: Some Summary Statistics Based on Information from Wainer (1983) and Oderwald (1983)

Statistics	Response to the Questions						Total
	(A)	(B)	(C)	(D)	(E)	Omit	
No. of Examinees	254,339	174,725	169,404	79,701	30,005	120,981	829,155
Proportion	.31	.21	.20	.10	.04	.15	
Correlation	.06	-.08	.52	-.09	-.32	-.15	
Mean Total Score	25.23	23.62	29.53	23.53	20.30	22.81	25.07
Standard Deviation	8.43	8.49	9.35	8.67	7.96	8.55	9.02

Illustrations of Keys

In the PSAT Item 44, the pyramid *ABCD* is a triangular pyramid and often called a regular tetrahedron. The pyramid *EFGHI* is a square pyramid and also called a regular or complete quadrilateral pyramid.

For the purpose of the illustrations, it is assumed that the faces of the triangular pyramid are colored with yellow-*ABC*, red-*ACD*, and blue-*ADB*; and the triangular base is colored with black-*BCD*. It is also assumed that the faces of the square pyramid are colored with yellow-*EFG*, orange-*EGH*, purple-*EHI*, and green-*EIF*; and the square base is colored with gray-*FGHI*.

The same yellow color was used for face *ABC* and face *EFG* because face *ABC* will be placed on face *EFG* so that the vertices of the triangles coincide. Face *ABC* and face *EFG* are triangular figures of the same size and shape. In geometry, when one plane figure fits exactly over another figure, then the two figures are said to coincide (Jurgensen et al., 1965). Note that face *ABC* and face *EFG* form triangles that are congruent. Therefore, mathematically, $A \equiv E$, $B \equiv F$, and $C \equiv G$, where = can be used instead of ≡.

A good starting point of a new solution is to visualize the situation from the right hand side that is parallel to line *BC* and line *FG*. Once *ABC* is onto *EFG*, because $B \equiv F$, and $C \equiv G$, all of these points constitute a center point of Figure 2 (say *O*). The lines from the center to *A* and from the center to *D* are the slant height of the triangular pyramid that is $\sqrt{3}/2$ of the unit length of the equilateral triangle. Line *AD* is the unit length. The lines from the center to *E* and from *E* to *H* on Figure 2 are also the slant height of the one side of the square pyramid. Line *GH* is the unit length. Because the altitude of the triangular pyramid, $\sqrt{2/3}$ of the unit length, is greater than that of the square pyramid, $1/\sqrt{2}$, and the projected angle of *HOD* in Figure 2 is 15:79°, the resulting solid is complicated and multilateral in its shape.

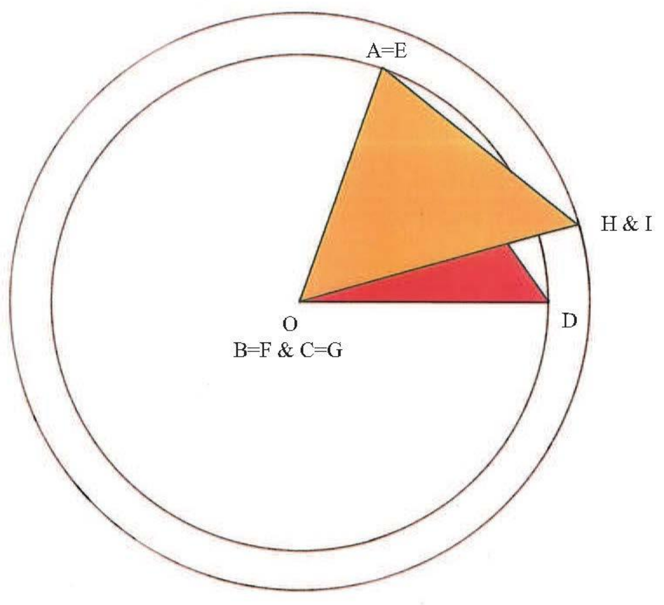


Figure 2: Projection of the resulting solid from the viewing position of the right hand side.

Now, let's construct images of the resulting solid from six different viewing points. Figure 3 contains the six images. In Figure 3, all images are the results from orthogonal or parallel projections to base *BCD* and the line between *D* and the middle point of line *BC*. Such a line is the slant height of the triangular pyramid. Note that Figure 3 and other figures in the manuscript contain proportionally correct images.

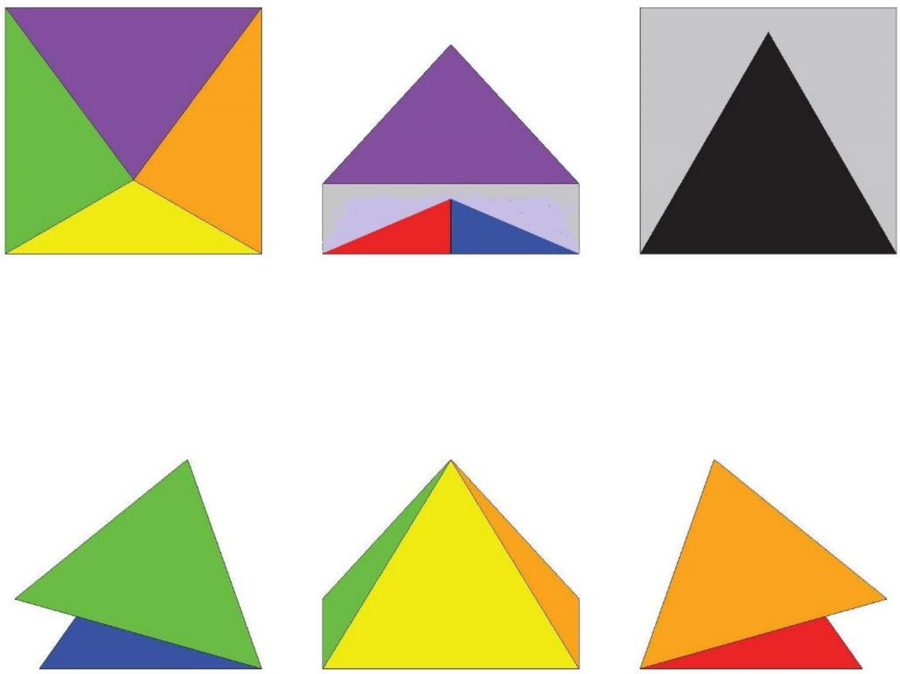


Figure 3: Images of the resulting solid from the top, back, and bottom viewing positions in the first row and the left, front, and right viewing positions in the second row sitting on the triangular base.

Because all eight colors from both the triangular pyramid and the square pyramid can be seen in Figure 3, the number of exposed faces of the resulting solid is eight; and accordingly the key of the question should be '(D) Eight.' It is now possible to make all eight colors are visible by the resulting solid standing with vertex *D* and the lateral edge *HI*, some previous preposterous arguments along with the constitutive meaning of "exposed" will no longer have any standing ground (e.g., Antonick 2013).

Only by changing portions of the stem of the question, for example, by replacing "*EFG*" with "*EGF*," the resulting solid will have a different number of so-called exposed faces. Assuming now that *A*≡*E*, *B*≡*G*, and *C*≡*F*, then the resulting solid can be illustrated in Figure 4. Note that in Figure 4, all images are the results from orthogonal or parallel projections to base *FGHI* base and the line *GH*.

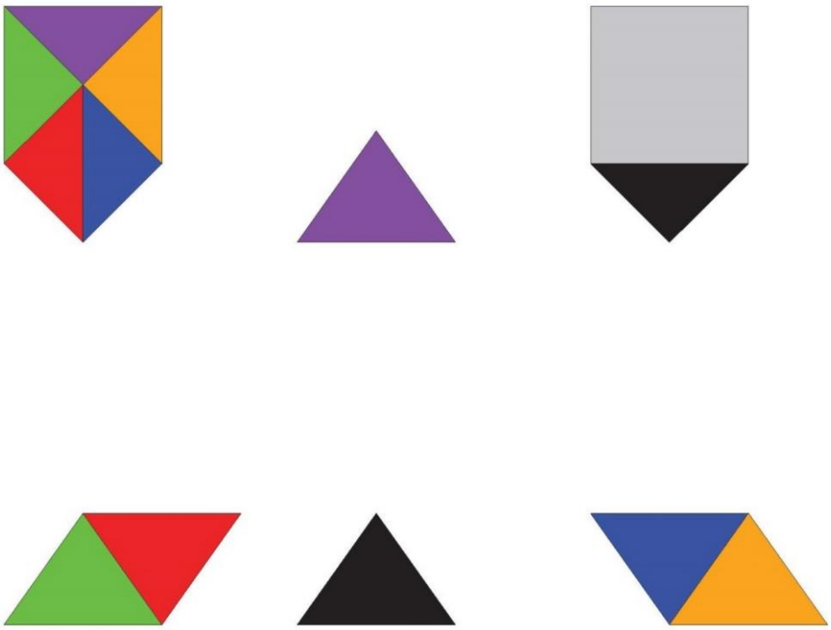


Figure 4: Images of the resulting solid with "EGF" from the top, back, and bottom viewing positions in the first row and the left, front, and right viewing positions in the second row sitting on the square base.

There are five colors visible from Figure 5. The challenge to the original intended key '(C) Seven' with the new key '(A) Five' was mentioned in New York Times on March 17, 1981 (Fiske 1981; see also Antonick 2013); and the proof that there are five faces in the resulting solid was given in Wainer (1981, p. 21). The resulting solid is in fact an oblique prism when rotated to the one where either the frontal or rear projection in Figure 5 becomes the base of the equiangular, equilateral triangular prism. Two faces *ADB* and *EGH* are coplanar and become just one parallelogram face. The resulting face of (*E*≡*A*)*D*(*G*≡*B*)*H* is colored with orange in Figure 5. Similarly two faces *ACD* and *EIF* are coplanar and become just one parallelogram face. The resulting face of (*E*≡*A*)*D*(*F*≡*C*)*I* is colored with green in Figure 5.

Discussion

If the illustrations made the readers to think '(D) Eight' may be another correct answer, there are several issues to ponder over. Because the ETS and the College Board decided to treat possibly incorrect answers to be scored correctly, for the given PSAT about 51 per cent of the test takers (i.e., 423,743 out of 829,155 examinees) were mistakenly rewarded if such a conjecture is valid. In fact, the PSAT is the National Merit Scholarship Qualifying Test (NMSQ) used as a qualifying test for the National Merit Scholarship Program. We think some people might have been somehow either positively or negatively impacted by the pyramid question.

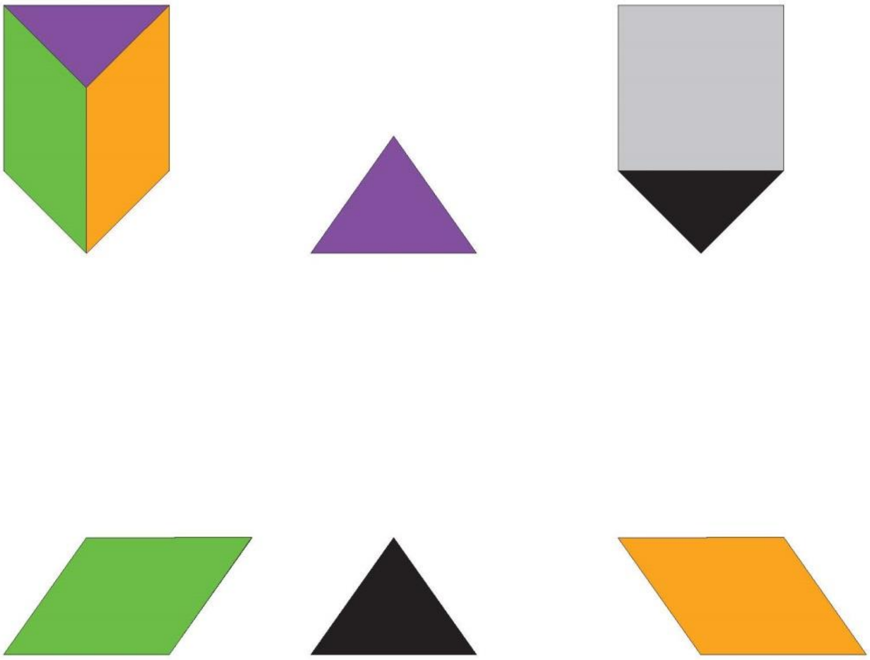


Figure 5: Images of the better resulting solid with "EGF" from the top, back, and bottom viewing positions in the first row and the left, front, and right viewing positions in the second row sitting on the square base.

Observing positive values of point-biserial correlation for the two options, as treated by the ETS and the College Board, can be seen as a good justification for the final scoring of the pyramid question. It should be noted, however, that in conjunction with the classical test theory framework, positive correlation can be obtained by arbitrarily selecting portions of the examinee group (e.g., a low performing examinee group will produce a positive correlation value for '(D) Eight' choice).

In the multiple choice question format the examinees are free to guess (although possibly they have received a direction that contains a warning about correction for guessing), and obviously some will arrive to the choice that will be scored correctly. Whatever the scoring key is, there are examinees who arrived to the scoring key by lucky guessing and via a wrong reason, respectively. Because inscribed, circumscribed, as well as tangential polygons and solids are treated in geometry, some examinees might have obtained the answer by employing incorrectly inscribed solids.

When hearing the statement that "some of the examinees who encountered the pyramid question must have reached the answer illustrated in this manuscript and selected '(D) Eight' to be the correct answer," one of the authors said that those people were possibly creative but really unlucky ones. We do not know how many of the about 10 per cent of the test takers (i.e., 79,701 out of 829,155 examinees) as well as those of the earlier SAT takers who encountered and selected the option by exercising the logic described in the current manuscript.

Lastly, in our opinion, the resulting solid may form both a triangular pyramid by vertices of *ABCD* and a regular quadrilateral pyramid by vertices of *EFGHI*. Nevertheless, we do not believe the pyramid question is a good, valid question. Note also that validity of the inference from test scores may not be judged by a single item. We look forward to hearing others' suggestions.

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