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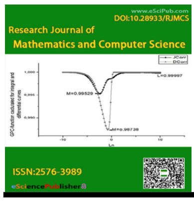

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Double Lipschitz Stability for Nonlinearly neutral Differential Systems with Multiple Delay

Huo Ran[1], Wang Xiaoli[2]*

[1]College of Science Inner Mongolia Agricultural University, Huhhot, 010018,China.
[2]College of Statistics and Mathematics, Inner Mongolia University of Finance and Economics, Huhhot, 010070, China.

ABSTRACT

In this paper, first we proves a class of time-delay differential inequalities. And then, the nonlinear differential systems with time delay are considered and it is obtained that the trivial solution of the nonlinear systems with time delay has uniform stability and uniform exponential Lipschitz asymptotic stability with respect to partial variable.

Keyword: Nonlinear systems; Partial variable stability; Lipschitz asymptotic stability; Differential inequality

Mathematics Subject Classification 2010:37C75

***Correspondence to Author:**
Wang Xiaoli
College of Statistics and Mathematics, Inner Mongolia University of Finance and Economics, Huhhot, 010070, China.

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1. INTRODUCTION

In 1892, Lyapunov, a Russian mathematician, mechanician and physicist, first proposed the definition of the stability of motion. He gave the general research methods in his PhD thesis "The general problem of the stability of motion", which established the foundation of the stability theory. With the development of science, Lyapunov theory is being expanded and developed constantly and is applied extensively in delay differential equations. Since it is not necessary to study all variables considering the actual need, it is of practical significance that studying the partial stability of differential equations. Vorotnikov. V. I. ([1,2]) considered the following system

$$\begin{cases} \frac{dy}{dt} = A(t)y + B(t)z + Y(t,y,z) \\ \frac{dz}{dt} = C(t)y + D(t)z + Z(t,y,z) \end{cases} \quad (1)$$

and studied the double stability as $\|y\| + \|z\| \to 0$ and $\frac{\|Y(t,y,z)\| + \|Z(t,y,z)\|}{\|y\| + \|z\|} \to 0$

In this paper the authors consider a new class of the nonlinearly perturbed nonlinear differential systems with time-delay

$$\frac{dx}{dt} = A(t)x + f(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau))\mathrm{d}s), \qquad (2),$$

where $x \in R^n$, $y = col(x_1, x_2, \cdots, x_m)$, $z = col(x_{m+1}, x_{m+2}, \cdots, x_n)$, $x = col(y, z)$,

$f : R \times R^n \times C([-r,0], R^n) \times R^n \mapsto R^n$, $f(t,0,0,0) \equiv 0$, $h : R \times R^n \times C([-r,0], R^n) \mapsto R^n$,

τ is a nonnegative constant.

It is obvious that the above system is a generalization of the systems in [1]-[5]. The aim of this paper is to investigate the double stability of time-delay differential equations, including Uniform stability and Uniform Lipschitz stability. The authors use the method of differential inequalities with time-delay and integral inequalities to establish double stability criteria.

2. PRELIMINARIES

Consider the following system:

$$\frac{dx}{dt} = f(t, x(t), x(t-\tau), \dot{x}(t-\tau)) \qquad (3)$$

where $x \in R^n$, $x \in R^n$, $y = col(x_1, x_2, \cdots, x_m)$, $z = col(x_{m+1}, x_{m+2}, \cdots, x_n)$, $x = col(y, z)$, $f(t,0,0) \equiv 0$,

τ is a nonnegative constant. Let $\phi(t)$ be a continuous function, for $\forall t \in E_{t_0} = [t_0 - \tau, t_0]$.

Definition1 The trivial solution of system (3) has uniform stability and exponential asymptotic stability with respect to y if, for $\forall \varepsilon > 0$, $\forall t_0 \in I$, $\exists \delta(\varepsilon) > 0$, and $\lambda > 0$, when $\|\varphi\| + \|\dot{\varphi}\| < \delta$ (for $\forall t \in E_{t_0}$), such that

$$\|y(t;t_0,\phi)\| + \|\dot{y}(t;t_0,\phi)\| < \varepsilon \exp(-\lambda(t-t_0)), ?\forall t \ge t_0.$$

Definition2 The trivial solution of system (3) has Lipschitz stability with respect to y if, there exists constants $M(t_0) > 0$ and $\delta(t_0) > 0$, when $\|\varphi\| + \|\dot{\varphi}\| < \delta$ (for $\forall t \in E_{t_0}$), such that

$$\|y(t;t_0,\phi)\| + \|\dot{y}(t;t_0,\phi)\| \le M(t_0)(\|\phi\| + \|\dot{\phi}\|), ?\forall t \ge t_0 \ge 0.$$

Definition3 The trivial solution of system (3) has equi-exponential Lipschitz asymptotic stability with respect to y if, there exists $\lambda > 0$, $K(t_0) > 0$ and $\delta(t_0) > 0$, when $\|\varphi\| + \|\dot{\varphi}\| < \delta$ (for $\forall t \in E_{t_0}$), such that

$$\| y(t;t_0,\phi)\| + \| \dot{y}(t;t_0,\phi)\| \le K(t_0)(\|\phi\| + \|\dot{\phi}\|)\exp(-\lambda(t-t_0)),?\forall t \ge t_0 \ge 0.$$

Definition4 The trivial solution of system (3) has uniform exponential Lipschitz asymptotic stability with respect to y if, K and $\delta > 0$ in definition 3 are dependent of t_0.

Lemma 1[6] Assume the homogeneous system

$$\begin{cases} \dfrac{dy}{dt} = B(t)y + C(t)z \\ \dfrac{dz}{dt} = D(t)y + E(t)z \end{cases} \qquad (4)$$

if the trivial solution of (4) has uniform stability and exponential asymptotic stability with respect to y, which is obtained under the condition of theorem, satisfies following conditions:

$$\| y \| \le V(t,x) \le M \| x \|,?\dot{V}|_{(4)} \le -\alpha V(t,x),$$

Lemma 2[7] Consider the following inequality:

$$\dot{x}_i(t) \le f_i(t)[-r_i x_i(t) + h_i^{(1)}(x_t)x_t + \int_{-\infty}^{t} h_i^{(2)}(t-s,x(s))x(t)e^{-\varepsilon(t-s)}\mathrm{d}s] \qquad (5)$$

where $f_i(t) \in C[R,R^+]$ and $f_i(t) \ge \gamma = const > 0$, $r_i = const > 0?(i=1,2,\cdots,m)$,

$h_i^{(1)}(\cdot)$, $h_i^{(2)}(t-\theta,\cdot)$ are nonnegative and not monotone decreasing for "$\cdot$", $x(\theta) = \max\limits_{1\le j\le n}(x_j(\theta))$,

$x_t = \max\limits_{1\le j\le n}(\sup\limits_{t-\tau<\theta<t} x_j(\theta))$, $\tau = const > 0$.

Assume $x_i(t)$ be nonnegative continuous on R_+, for all $t \ge t_0$ (5) is satisfied, if $\exists K = const$, the following inequality holds:

$$h_i^{(1)}(K) + \int_0^{+\infty} h_i^{(2)}(s,K)ds < r_i,$$

when $M \overset{\Delta}{=} \max\limits_{1\le j\le n}(\sup\limits_{t_0-\tau\le\theta\le t_0} x_j(\theta)) < K$, we have following result for all $t \ge t_0$, and $\lambda > 0$

$$x_i(t) \le M\exp(-\lambda(t-t_0))?\ (i=1,2,\cdots,m)$$

Lemma 3[14] Consider the following inequality:

$$\dot{x}_i(t) \le f_i(t)[-r_i x_i(t) + h_i^{(1)}(x_t)x_t^{\alpha_i} + \int_{-\infty}^{t} h_i^{(2)}(t-s,x(s))x^{\beta_i}(t)e^{-\varepsilon(t-s)}\mathrm{d}s] \quad (6)$$

where $f_i(t) \in C[R,R^+]$ and $f_i(t) \ge \gamma = const > 0$, $r_i = const > 0?(i=1,2,\cdots,m)$,

$h_i^{(1)}(\cdot)$, $h_i^{(2)}(t-\theta,\cdot)$ are nonnegative and not monotone decreasing for "$\cdot$", $\alpha_i, \beta_i \ge 1$,

$x(\theta) = \max\limits_{1\le j\le n}(x_j(\theta))$, $x_t = \max\limits_{1\le j\le n}(\sup\limits_{t-\tau<\theta<t} x_j(\theta))$, $\tau = const > 0$, $\alpha = \max(\alpha_i, \beta_i)$.

Assume $x_i(t)$ be nonnegative continuous on R_+, for all $t \ge t_0$ (6) is satisfied, if $\exists K = const$, the following inequality holds:

$$h_i^{(1)}(K) + \int_0^{+\infty} h_i^{(2)}(s,K)ds < r_i,?\ (8)?\alpha K^{1-\frac{1}{\alpha}} < 1$$

when $M^{\alpha} \overset{\Delta}{=} \max\limits_{1\le j\le n}(\sup\limits_{t_0-\tau\le\theta\le t_0} x_j(\theta)) < K$, we have following result for all $t \ge t_0$, and $\lambda > 0$

$$x_i(t) \le M\exp(-\lambda(t-t_0))?\ (i=1,2,\cdots,m)$$

3. MAIN RESULTS

Consider the following system

$$\begin{cases} \dfrac{dy}{dt} = B(t)y + C(t)z + Y(s, y(s), z(s), \int_0^t h_1(s, y(s), z(s), y(s-\tau), \dot{z}(s-\tau))\mathrm{d}s) \\ \dfrac{dz}{dt} = D(t)y + E(t)z + Z(s, y(s), z(s), \int_0^t h_2(s, y(s), z(s), y(s-\tau), \dot{z}(s-\tau))\mathrm{d}s) \end{cases} \quad (12)$$

where $\tau \ge 0$ is a constant, initial condition is

$$x(t) = \phi(t)?,?t_0 - \tau \le t \le t_0,$$

$B(t)$ is an $m \times m$ matrix, $Y(s, y(s), z(s), \int_0^t h_1(s, y(s), z(s), y(s-\tau), z(s-\tau))\mathrm{d}s)$ is an $m \times 1$ matrix, $Z(s, y(s), z(s), \int_0^t h_2(s, y(s), z(s), y(s-\tau), z(s-\tau))\mathrm{d}s)$ is an $(n-m)\times 1$ matrix, they are all continuous for $t \in I$ and satisfy the condition of existence and uniqueness theorem.

the homogeneous system of (12) is

$$\begin{cases} \dfrac{dy}{dt} = B(t)y + C(t)z \\ \dfrac{dz}{dt} = D(t)y + E(t)z \end{cases} \quad (12)^*$$

Theorem If (12) satisfies the following conditions:

i) $Y(t,0,0,0,0,0) \equiv Y(t,0,z,0,z(t-\tau),0) \equiv 0$;

ii) $Z(t,0,0,0,0,0) \equiv Z(t,0,z,0,z(t-\tau),0) \equiv 0$;

iii)
$$\begin{aligned} &\| Y(t, y, z, y(t-\tau), z(t-\tau), \int_0^t h_1(s, y, z, y(s-\tau), \dot{z}(s-\tau))\mathrm{d}s) \| \\ &\le r\|y\| + h^{(1)}(\|y(t-\tau)\|^\alpha)(\|z(t-\tau)\|^\alpha + \|\dot{z}(t-\tau)\|^\alpha) \\ &+ \int_{-\infty}^t h^{(2)}(t-s, \|y(s)\|^\alpha)(\|z(s)\|^\alpha + \|\dot{z}(s)\|^\alpha)e^{-\varepsilon(t-s)}\mathrm{d}s) \end{aligned}$$

iv)
$$\begin{aligned} &\| Z(t, y, z, y(t-\tau), z(t-\tau), \int_0^t h_2(s, y, z, y(s-\tau), \dot{z}(s-\tau))\mathrm{d}s) \| \\ &\le r\|y\| + h^{(1)}(\|y(t-\tau)\|^\alpha)(\|z(t-\tau)\|^\alpha + \|\dot{z}(t-\tau)\|^\alpha) \\ &+ \int_{-\infty}^t h^{(2)}(t-s, \|y(s)\|^\alpha)(\|z(s)\|^\alpha + \|\dot{z}(s)\|^\alpha)e^{-\varepsilon(t-s)}\mathrm{d}s) \end{aligned}$$

where $h^{(1)}(\cdot)$, $h^{(2)}(t-\theta,\cdot)$ are nonnegative and not monotone decreasing for $\cdot$, $\alpha \ge 1$, and

$$h^{(1)}(K) + \int_0^{+\infty} h^{(2)}(s,K)ds < r,?\ \alpha K^{1-\frac{1}{\alpha}} < 1$$

then the trivial solution of system (12) has uniform exponential Lipschitz asymptotic stability with respect to y, if the trivial solution of system (12)* has uniform stability and exponential asymptotic stability with respect to y.

Proof By the lemma 1, The V-Lyapunov function of (12)*, which is obtained under the condition of theorem, satisfies following conditions:

$$\|y\| \le V(t,x) \le M\|x\|,?\dot{V}|_{(11)^*} \le -\alpha V(t,x), \quad (13)$$

$| V(t,x'')-V(t,x') |\leq M \| x''-x' \|$? (14), $(a, M = const > 0)$, $for ? \geq 0, \| x \| < \infty$ Derivative the V-Lyapunov function $V(t,x)$ along (12), we get

$$\dot{V}_{(12)} \leq -aV(t,x) + R(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s),$$

where

$$R(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau))\mathrm{d}s) = \left\langle \frac{\partial V}{\partial x}, X^*(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s) \right\rangle$$

$$X^*(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s)$$

$$= \{Y(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s), Z(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s)\}^T.$$

here <,> is the notation of inner product.

From condition of theorem and (14), when $t \geq t_0$ we have

$$| R(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s) |$$

$$\leq M[r \| y(t) \| + h^{(1)}(\| y(t-\tau) \|^\alpha)(\| z(t-\tau) \|^\alpha + \| \dot{z}(t-\tau) \|^\alpha)$$
$$+ \int_{-\infty}^t h^{(2)}(t-s, \| y(s) \|^\alpha)(\| z(s) \|^\alpha + \| \dot{z}(s) \|^\alpha) e^{-\varepsilon(t-s)}\mathrm{d}s)].$$

By the first inequality of (13)，the above can be expressed as follow:

$$| R(t, x(t), x(t-\tau), \int_0^t h(s, x(s), x(s-\tau), \dot{x}(s-\tau))\mathrm{d}s) |$$

$$\leq M[rV(t,x) + h^{(1)}(V^\alpha(t,x))V^\alpha(t,x) + \int_{-\infty}^t h^{(2)}(t-s, V^\alpha(t,x))V^\alpha(t,x) e^{-\varepsilon(t-s)}\mathrm{d}s)].$$

then there exists $K > 0$ such that when $t \geq t_0$ and $\sup_{t_0-\tau\leq\sigma\leq t_0} \{V(\sigma, x(\sigma, t_0, \phi))\} < K$,we get

$$\dot{V}|_{(12)}(t,x) \leq -aV(t,x) + M[rV(t,x) + h^{(1)}(V^\alpha(t,x))V^\alpha(t,x) + \int_{-\infty}^t h^{(2)}(t-s, V^\alpha(t,x))V^\alpha(t,x) e^{-\varepsilon(t-s)}\mathrm{d}s]$$

$$= M[-(\frac{a}{M} - r)V(t,x) + h^{(1)}(V^\alpha(t,x))V^\alpha(t,x) + \int_{-\infty}^t h^{(2)}(t-s, V^\alpha(t,x))V^\alpha(t,x) e^{-\varepsilon(t-s)}\mathrm{d}s].$$

here select the appropriate small constant r such that

$$r' = \frac{a}{M} - r > 0 ? and \ h^{(1)}(K) + \int_0^{+\infty} h^{(2)}(s, K)\mathrm{d}s < r',$$

hence by the lemma 2, there exists $\lambda>0$ such that for all $t \geq t_0$ we have

$$V(t,x) < \sup_{t_0-\tau\leq\sigma\leq t_0} \{V(\sigma, x(\sigma, t_0, \phi))\} \exp(-\lambda(t-t_0)) \quad (15)$$

For any solution of (11), from the inequality (14) and the first inequality of (12) we obtain

$$\| y(t) \| + \| \dot{y}(t) \| \leq V(t,x) \leq M(\| \phi \| + \| \dot{\phi} \|) \exp(-\lambda(t-t_0)).$$

According to the proof of the theorem in [5] we get $\|x\|<\varepsilon$, hence we obtain that the trivial solution of system (12) has uniform stability and uniform exponential Lipschitz asymptotic stability with respect to y .

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